

THE ANOMALY FLOW AND THE FU-YAU EQUATION¹

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Abstract

The anomaly flow is shown to converge when restricted to toric fibrations with the Fu-Yau ansatz, for both positive and negative values of the slope parameter α' . This implies both results of Fu and Yau on the existence of solutions for Strominger systems, which they proved using different methods depending on the sign of α' . It is also the first case where the anomaly flow can even be shown to exist for all time. This is in itself of particular interest from the point of view of the theory of fully nonlinear partial differential equations, as the elliptic terms in the flow are not concave.

1 Introduction

The Strominger system [27] is a system of equations for supersymmetric compactifications of the heterotic string, which is less restrictive than the Ricci-flat conditions originally proposed by Candelas, Horowitz, Strominger, and Witten [1]. More specifically, it is a system of three equations for a Hermitian metric ω on a 3-fold Y with $c_1(Y) = 0$ and a Hermitian metric $H_{\bar{\alpha}\beta}$ on a holomorphic vector bundle over Y . The first equation is the condition that the Chern unitary connection of $H_{\bar{\alpha}\beta}$ be Yang-Mills, the second equation is the famous Green-Schwarz anomaly cancellation equation from superstring theory, and the third is a torsion constraint on χ . Thus the first equation is well-known, and much of the novelty and difficulty in Strominger systems resides in the last two equations. Anomaly flows were proposed by the authors in [22] as a natural way of finding solutions to the Green-Schwarz anomaly cancellation equation, while implementing the torsion constraints. However, the anomaly flows themselves present many new challenges. To begin with, the flows of the metric χ are not given directly, but have to be recaptured from the flows of $\|\Omega_Y\|_\chi \chi^2$, where Ω_Y is a fixed nowhere vanishing holomorphic $(3,0)$ -form. This difficulty was recently overcome in [23], but many other difficulties remain, including the presence of the dilaton field $\|\Omega\|_\chi$ and especially of quadratic terms in the curvature tensor. In particular, unless the slope parameter α' is zero [23], even criteria for the long-time existence of the flows are not yet known.

While a complete analysis of the anomaly flow in full generality appears to be out of reach at the present time, there is one geometric setting of considerable interest which

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should serve as a good guide for future studies. This is the case of the toric fibrations $\pi : Y \rightarrow X$ over Calabi-Yau surfaces $(X, \hat{\omega})$ constructed by Goldstein and Prokushkin [14]. It has been shown by Fu and Yau [11] that, by choosing the vector bundle over Y to be the pull-back $\pi^*(E)$ of a stable bundle E over $(X, \hat{\omega})$ with zero slope and Hermitian-Yang-Mills metric $H_{\bar{\alpha}\beta}$, the Strominger system for $(Y, \pi^*(E))$ reduces to a single non-linear elliptic partial differential equation for a scalar function on X that they can solve. This suggests that Goldstein-Prokushkin fibrations may be a good testing ground for the anomaly flow.

The main result of our paper is the following:

Theorem 1 *Let $\pi : Y \rightarrow X$ be a Goldstein-Prokushkin fibration over a Calabi-Yau surface $(X, \hat{\omega})$ with Ricci-flat metric $\hat{\omega}$, and $E \rightarrow (X, \hat{\omega})$ a stable vector bundle with zero slope and Hermitian-Yang-Mills metric $H_{\bar{\alpha}\beta}$. Let θ be the $(1, 0)$ -form on Y described in §2.1 below. Assume that the cohomological condition $\int_X \mu = 0$ is satisfied, where μ is defined in (2.24). Consider the anomaly flow given by (2.6) below, with χ given by $\chi = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}$, and initial data $\chi(0) = \pi^*(M\hat{\omega}) + i\theta \wedge \bar{\theta}$, where M is a positive constant. Then there exists $M_0 \gg 1$ such that, for all $M \geq M_0$, the flow exists for all time, and converges to a smooth Hermitian metric ω_∞ , with $(\omega_\infty, \pi^*(H))$ a solution of the Strominger system for $(Y, \pi^*(E))$.*

In particular, the theorem recaptures the results of Fu and Yau in [11, 12], where they proved the existence of solutions of the Strominger system on Goldstein-Prokushkin fibrations satisfying the cohomological condition $\int_X \mu = 0$, when $\alpha' > 0$ and $\alpha' < 0$ respectively. But perhaps even more important is that the theorem provides a strong evidence that the anomaly flow should be a viable method of solution for Strominger systems. In general, there are infinitely many parabolic flows whose stationary points would satisfy the same given elliptic equation. It is well-known in PDE theory that, analytically, some parabolic flows are better behaved than others, for example, those with concavity properties as functions of the Hessian of the unknown (see e.g. [3, 16, 19]). However, in the case of Strominger systems, the parabolic equation defined by the anomaly flow is already dictated by the simultaneous requirement that the stationary points satisfy both the Green-Schwarz anomaly cancellation requirement and the torsion constraints, and there is no further flexibility in selecting the flow. It is then a great reassurance to find out that it is analytically well-behaved. Furthermore, as we shall also see below in §2.2, it turns out that this special case of the Goldstein-Prokushkin fibration retains enough of the original problem for it to be a likely good model for the general case.

Restricted to a Goldstein-Prokushkin fibration, the anomaly flow becomes equivalent to the following flow for a metric $\omega = ig_{k\bar{j}} dz^j \wedge d\bar{z}^k$ on a Calabi-Yau surface X , equipped with a nowhere vanishing holomorphic $(2, 0)$ -form Ω ,

$$\partial_t \omega = -\frac{1}{2\|\Omega\|_\omega} \left(\frac{R}{2} - |T|^2 - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) + 2\alpha' \frac{i\partial\bar{\partial}(\|\Omega\|_\omega \rho)}{\omega^2} - 2\frac{\mu}{\omega^2} \right) \omega \quad (1.1)$$

where $\sigma_2(\Phi) = \Phi \wedge \Phi \omega^{-2}$ is the usual determinant of a real $(1, 1)$ -form Φ , relative to the metric ω . The expression $|T|^2$ is the norm of the torsion of ω defined in (2.39) below. Thus the theorem that we shall actually prove is the following

Theorem 2 *Let $(X, \hat{\omega})$ be a Calabi-Yau surface, equipped with a Ricci-flat metric $\hat{\omega}$ and a nowhere vanishing holomorphic $(2, 0)$ -form Ω normalized by $\|\Omega\|_{\hat{\omega}} = 1$. Let α' be a non-zero real number, and let ρ and μ be smooth real $(1, 1)$ and $(2, 2)$ -forms respectively, with μ satisfying the integrability condition*

$$\int_X \mu = 0. \quad (1.2)$$

Consider the flow (1.1), with an initial metric given by $\omega(0) = M \hat{\omega}$, where M is a constant. Then there exists M_0 large enough so that, for all $M \geq M_0$, the flow (1.1) exists for all time, and converges exponentially fast to a metric ω_∞ satisfying the Fu-Yau equation

$$i\partial\bar{\partial}(\omega_\infty - \alpha'\|\Omega\|_{\omega_\infty}\rho) - \frac{\alpha'}{8}\text{Ric}_{\omega_\infty} \wedge \text{Ric}_{\omega_\infty} + \mu = 0, \quad (1.3)$$

and the normalization $\int_X \|\Omega\|_{\omega_\infty} \frac{\omega_\infty^2}{2!} = M$.

Since the flow is conformal, it can be rewritten as a flow of the conformal factor $\omega = e^u \hat{\omega}$,

$$\partial_t u = \frac{1}{2} \left(\Delta_{\hat{\omega}} u + \alpha' e^{-u} \hat{\sigma}_2(i\partial\bar{\partial}u) - 2\alpha' e^{-u} \frac{i\partial\bar{\partial}(e^{-u}\rho)}{\hat{\omega}^2} + |Du|_{\hat{\omega}}^2 + e^{-u} \tilde{\mu} \right) \quad (1.4)$$

where $\tilde{\mu} = 2\mu \hat{\omega}^{-2}$ is a time-independent scalar function, and both the Laplacian $\Delta_{\hat{\omega}}$ and the determinant $\hat{\sigma}_2$ are written with respect to the fixed metric $\hat{\omega}$.

Setting the right hand side to 0 gives the equation solved by Fu and Yau [11, 12], so the anomaly flow is indeed a parabolic version of the Fu-Yau equation. Moreover, the equation (1.4) can be rewritten in the form

$$2\alpha' e^u \partial_t u = \frac{(e^u \hat{\omega} + e^{-u} \rho + \alpha' i\partial\bar{\partial}u)^2}{\hat{\omega}^2} + w(\mu, \rho, u, Du), \quad (1.5)$$

and it is a parabolic complex Monge-Ampère type equation. However, unlike the Kähler-Ricci flow where the elliptic term is $\log \det(g_{i\bar{j}} + u_{i\bar{j}})$, the equation (1.5) has none of the desirable concavity properties of elliptic and parabolic equations. In particular, none of the techniques used in [11, 12] for the elliptic case (as well as the ones for the more general equations in [24, 25]) can be adapted here besides the Moser iteration technique for the C^0 estimate. We shall see below that the proof of Theorem 2 relies instead, in an essential manner, on the geometric formulation of (1.1), and that the estimates are obtained using the metric which evolves with the flow.

Various geometric flows have been studied in non-Kähler complex geometry (see e.g. [4, 13, 28, 29, 30, 31, 32] and references therein). The main difficulty in studying (1.1) is that it is quadratic in the Ricci curvature. This creates substantial problems in applying known techniques to try and obtain estimates on the torsion, curvature, and derivatives of curvature. To overcome these issues, we first start the flow with a metric with vanishing Ricci curvature and torsion, and the objective is to show that for suitably large normalization of the initial metric, we can prevent the terms which are nonlinear in curvature from growing too large and dominating the behavior of the flow. Proposition 4 in §7 shows exactly how the estimates on Ricci curvature and torsion depend on the normalization of the initial metric.

The paper is organized as follows. In Section §2, we provide the background on Goldstein-Prokushkin fibrations, and show how to reduce the anomaly flow in this case to the flow (1.1) for metrics on a Calabi-Yau surface. Sections §3-§6 are devoted to successive estimates: the uniform boundedness of the metrics ω in §3, the estimates for the torsion in Section §4, the estimates for the curvature, and higher order derivatives of both the torsion and the curvature in Sections §5-§6. Finally, long time existence is shown in Section §7 and the convergence of the flow is proved in Section §8.

2 Anomaly flows on Goldstein-Prokushkin fibrations

Let Y be a compact 3-fold, equipped with a nowhere vanishing holomorphic $(3, 0)$ -form Ω_Y , and a complex vector bundle $E_Y \rightarrow Y$. The Strominger system is the following system of equations for a Hermitian metric χ on Y and a Hermitian metric $H_{\bar{\alpha}\beta}$ on E_Y ,

$$F^{2,0} = F^{2,0} = 0, \quad \chi^2 \wedge F = 0 \quad (2.1)$$

$$i\partial\bar{\partial}\chi - \frac{\alpha'}{4}\text{Tr}(Rm(\chi) \wedge Rm(\chi) - F(H) \wedge F(H)) = 0 \quad (2.2)$$

$$d^\dagger \chi = i(\partial - \bar{\partial}) \log \|\Omega\|_\chi. \quad (2.3)$$

Here $Rm(\chi)$ is the Riemann curvature tensor of the Chern unitary connection of χ , viewed as a $(1, 1)$ -form valued in the bundle $End(T^{1,0}(Y))$ of endomorphisms of $T^{1,0}(Y)$. Similarly, $F(H)$ is the curvature of the Hermitian metric H , viewed as a $(1, 1)$ -form valued in the bundle $End(E_Y)$ of endomorphisms of E_Y . The expression $\|\Omega_Y\|_\chi$ is the norm of Ω_Y with respect to the metric χ ,

$$\|\Omega_Y\|_\chi^2 = i\Omega_Y \wedge \overline{\Omega_Y} \chi^{-3}. \quad (2.4)$$

It was pointed out by Li and Yau [20] that the last condition above is equivalent to the condition that χ be conformally balanced,

$$d(\|\Omega_Y\|_\chi \chi^2) = 0. \quad (2.5)$$

In the same paper [20], Li and Yau obtained smooth solutions to the Strominger system by perturbing Kähler solutions. In [11, 12], Fu and Yau gave the first non-perturbative non-Kähler solution of the Strominger system. Solutions to the Strominger system have also been found on nilmanifolds and solvmanifolds [9, 10, 21, 33], hyperkähler fibrations over Riemann surfaces [6], and deformed and resolved conifolds [7, 8].

In [22], the authors pointed out that there is a natural flow, called there the anomaly flow, which preserves the conformally balanced condition, and whose stationary points are the solutions of the remaining equations in the Strominger system. This flow is a simultaneous flow for a metric χ on Y and a metric $H_{\bar{\alpha}\beta}$ on E_Y , and it is written out in full generality in [22], eq. (2.1) as

$$\begin{aligned}\partial_t H H^{-1} &= \Lambda_\chi F(H) \\ \partial_t (\|\Omega_Y\|_\chi \chi^2) &= i\partial\bar{\partial}\chi - \frac{\alpha'}{4} \text{Tr}(Rm(\chi) \wedge Rm(\chi) - F(H) \wedge F(H))\end{aligned}$$

where $\Lambda_\chi \psi = \chi^2 \wedge \psi \chi^{-3}$ is the Hodge operator on $(1,1)$ -forms ψ . However, for our purposes as we shall see shortly, we can assume that the metric $H_{\bar{\alpha}\beta}$ is known and fixed, and that $\Lambda_\chi F(H) = 0$ along the flow. Thus the curvature F is also known, and we need only consider the flow of the metric χ , which is

$$\partial_t (\|\Omega_Y\|_\chi \chi^2) = i\partial\bar{\partial}\chi - \frac{\alpha'}{4} \text{Tr}(Rm(\chi) \wedge Rm(\chi) - F \wedge F). \quad (2.6)$$

By Chern-Weil theory, the right hand side is a closed $(2,2)$ -form, so if the initial metric $\chi(0)$ is conformally balanced, which we assume from now on, the metric $\chi(t)$ will indeed stay conformally balanced for all t .

2.1 The Goldstein-Prokushkin fibration

We would like to restrict the anomaly flow from a general 3-fold Y to the special case of a Goldstein-Prokushkin fibration $\pi : Y \rightarrow X$. We begin by recalling the basic properties of Goldstein-Prokushkin fibrations that we need.

Let $(X, \hat{\omega})$ be a compact Calabi-Yau manifold of dimension 2, with $\hat{\omega}$ the Ricci-flat Kähler metric, and Ω a nowhere vanishing holomorphic $(2,0)$ -form, normalized so that

$$1 = \|\Omega\|_{\hat{\omega}}^2 = \Omega \wedge \bar{\Omega} \hat{\omega}^{-2}. \quad (2.7)$$

Let $\omega_1, \omega_2 \in 2\pi H^2(X, \mathbf{Z})$ be two $(1,1)$ -forms such that $\omega_1 \wedge \hat{\omega} = \omega_2 \wedge \hat{\omega} = 0$. From this data, Goldstein and Prokushkin [14] construct a compact 3-fold Y which is a toric fibration $\pi : Y \rightarrow X$ over X equipped with a $(1,0)$ form θ on Y satisfying

$$\bar{\partial}\theta = \pi^*(\omega_1 + i\omega_2) \quad \partial\theta = 0. \quad (2.8)$$

Furthermore, the $(3, 0)$ -form

$$\Omega_Y = \sqrt{3} \Omega \wedge \theta \quad (2.9)$$

is holomorphic and nowhere vanishing, and the $(1, 1)$ -form

$$\chi_0 = \pi^*(\hat{\omega}) + i\theta \wedge \bar{\theta} \quad (2.10)$$

is positive-definite on Y . Observe that

$$i\Omega_Y \wedge \overline{\Omega_Y} = 3\Omega \wedge \overline{\Omega} \wedge i\theta \wedge \bar{\theta} = \|\Omega\|_{\hat{\omega}}^2 (3\hat{\omega}^2 \wedge i\theta \wedge \bar{\theta}) = \|\Omega\|_{\hat{\omega}}^2 \chi_0^3. \quad (2.11)$$

Thus, defining the norm $\|\Omega_Y\|_{\chi}$ of the holomorphic form Ω_Y on Y with respect to a metric χ as in (2.4), we have $\|\Omega_Y\|_{\chi_0} = \|\Omega\|_{\hat{\omega}} = 1$. Consequently,

$$\|\Omega_Y\|_{\chi_0}^2 = \hat{\omega}^2 + 2i\hat{\omega} \wedge \theta \wedge \bar{\theta}. \quad (2.12)$$

This implies that $d(\|\Omega_Y\|_{\chi_0}^2) = 0$ by (2.8) and the fact that ω_1 and ω_2 wedged with $\hat{\omega}$ gives zero. Thus χ_0 is a conformally balanced metric on Y .

More generally, for any smooth function u on Y , introduce the following metrics ω_u and χ_u on the manifolds X and Y respectively,

$$\omega_u = e^u \hat{\omega}, \quad \chi_u = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}. \quad (2.13)$$

Then the same arguments that we just used show that

$$\|\Omega_Y\|_{\chi_u} = \|\Omega\|_{\omega_u} = e^{-u}, \quad (2.14)$$

and furthermore,

$$\|\Omega_Y\|_{\chi_u}^2 = \|\Omega\|_{\omega_u}^2 + 2i\hat{\omega} \wedge \theta \wedge \bar{\theta}. \quad (2.15)$$

This shows that $d(\|\Omega_Y\|_{\chi_u}^2) = 0$ since $d(\|\Omega\|_{\omega_u}^2)$ is the pull-back of a differential form of rank 5 defined on the 4-dimensional manifold X , and $2id(\hat{\omega} \wedge \theta \wedge \bar{\theta})$ is zero as before in view of (2.8). It follows that the metric χ_u is a conformally balanced metric on Y for any choice of u .

2.2 The Fu-Yau Ansatz

In [11, 12], Fu and Yau obtain a solution of the Strominger system in the following manner. Let $\pi : Y \rightarrow X$ be a Goldstein-Prokushkin fibration, constructed as described above from a Calabi-Yau surface $(X, \hat{\omega})$, equipped with two integer-valued harmonic $(1, 1)$ -forms $\omega_1/(2\pi)$ and $\omega_2/(2\pi)$.

Let $E_X \rightarrow X$ be a stable holomorphic vector bundle over X , with slope $\int_X c_1(E_X) \wedge \hat{\omega} = 0$. Then by the Donaldson-Uhlenbeck-Yau [5, 34] theorem, E_X admits a metric H_X

with respect to $\hat{\omega}$ satisfying the Hermitian-Yang-Mills equation $\hat{\omega} \wedge F(H_X) = 0$. Let $E = \pi^*(E_X) \rightarrow Y$ be the pull-back bundle over Y , and let $H = \pi^*(H_X)$. Since

$$\chi_u^2 \wedge F(H) = \pi^*(e^u \hat{\omega} \wedge F(H_X)) \wedge (e^u \hat{\omega} + 2i\theta \wedge \bar{\theta}) = 0, \quad (2.16)$$

it follows that H is Hermitian-Yang-Mills with respect to χ_u , for any u . Now recall that χ_u is conformally balanced for any u . This means that, if we look for a solution of the Strominger system under the form (Y, E) , equipped with the metrics χ_u and H , then the only equation which is left to solve is the Green-Schwarz anomaly cancellation equation (2.2), with the scalar function u defining the metric χ_u as the unknown.

The key property that allows this approach to work is that, for metrics of the form χ_u in a Goldstein-Prokushkin fibration, the equation on Y

$$i\partial\bar{\partial}\chi_u - \frac{\alpha'}{4}\text{Tr}(Rm(\chi_u) \wedge Rm(\chi_u) - F \wedge F) = 0 \quad (2.17)$$

descends to an equation on the base X . This was established by Fu and Yau [11] and a summary of their results is as follows.

First, the term $i\partial\bar{\partial}\chi_u$ is readily worked out, using the properties (2.8) of the form θ ,

$$i\partial\bar{\partial}\chi_u = i\partial\bar{\partial}\omega_u - \bar{\partial}\theta \wedge \partial\bar{\theta}. \quad (2.18)$$

Next, the quadratic term in the curvature tensor can be worked out to be (Proposition 8 in [11])

$$\text{Tr}(Rm(\chi_u) \wedge Rm(\chi_u)) = \text{Tr}(Rm(\omega_u) \wedge Rm(\omega_u)) + \partial\bar{\partial}(\|\Omega\|_{\omega_u} \text{Tr}(\bar{\partial}B \wedge \partial B^* \cdot \hat{\omega}^{-1})). \quad (2.19)$$

Here B is a $(1,0)$ -form depending on the data $(\hat{\omega}, \omega_1, \omega_2)$, which is only locally defined. However the full expression $\text{Tr}(\bar{\partial}B \wedge \partial B^* \cdot \hat{\omega}^{-1})$ is not only globally well-defined on Y , but it is the pull-back of a globally defined real $(1,1)$ -form ρ on X ,

$$\frac{1}{4}\text{Tr}(\bar{\partial}B \wedge \partial B^* \cdot \hat{\omega}^{-1}) = \pi^*(\rho). \quad (2.20)$$

On the other hand, from $\omega_u = e^u \hat{\omega}$, it follows that

$$Rm(\omega_u) = -\partial\bar{\partial}u \otimes I + Rm(\hat{\omega}) \quad (2.21)$$

and hence, in view of the fact that the metric $\hat{\omega}$ is Ricci-flat,

$$\text{Tr}(Rm(\omega_u) \wedge Rm(\omega_u)) = \text{Tr}(Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u. \quad (2.22)$$

Altogether, the Green-Schwarz anomaly cancellation equation can be written as the following equation on the base manifold X ,

$$0 = i\partial\bar{\partial}(\omega_u - \alpha'\|\Omega\|_{\omega_u}\rho) - \frac{\alpha'}{2}(\partial\bar{\partial}u) \wedge (\partial\bar{\partial}u) + \mu \quad (2.23)$$

where we have set

$$\mu = -\bar{\partial}\theta \wedge \partial\bar{\theta} - \frac{\alpha'}{4}\text{Tr}(Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) + \frac{\alpha'}{4}\text{Tr}(F(H_X) \wedge F(H_X)). \quad (2.24)$$

which is a well-defined $(2,2)$ -form on X . The equation (2.23) is the Fu-Yau equation. Clearly, a necessary condition for the existence of solutions is

$$\int_X \mu = 0 \quad (2.25)$$

This condition was shown to be sufficient by Fu and Yau in [11] for $\alpha' > 0$ and in [12] for $\alpha' < 0$. Examples of fibrations $\pi : Y \rightarrow X$ and vector bundles $E \rightarrow X$ satisfying $\int_X \mu = 0$ are exhibited in [11, 12].

2.3 Reduction of the anomaly flow by the Fu-Yau Ansatz

We consider now the anomaly flow (2.6) on a Goldstein-Prokushkin fibration $\pi : Y \rightarrow X$, equipped with the holomorphic $(3,0)$ -form Ω_Y and restricted to metrics of the form $\chi = \chi_u$. Recall that $F = F(\pi^*(H_X))$ is fixed.

We work out both sides of the flow (2.6) in this setting. Recall that we have set $\omega_u = e^u \hat{\omega}$ which is a metric on X . From the earlier equation (2.15) and the fact that the term $\hat{\omega} \wedge \theta \wedge \bar{\theta}$ is time-independent, it follows at once that

$$\partial_t(\|\Omega_Y\|_{\chi_u} \chi_u^2) = \partial_t(\|\Omega\|_{\omega_u} \omega_u^2). \quad (2.26)$$

On the other hand, the same formulas derived by Fu and Yau [11] for their reduction of the anomaly equation on Y to an equation on X and which we described in the previous section give

$$\begin{aligned} i\partial\bar{\partial}\chi_u - \frac{\alpha'}{4}\text{Tr}(Rm(\chi_u) \wedge Rm(\chi_u) - F \wedge F) \\ = i\partial\bar{\partial}(\omega_u - \alpha'\|\Omega\|_{\omega_u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu \end{aligned} \quad (2.27)$$

with μ the $(2,2)$ -form defined by (2.24). Thus the anomaly flow for Goldstein-Prokushkin fibrations is equivalent to the flow for metrics on X given by

$$\partial_t(\|\Omega\|_{\omega_u} \omega_u^2) = i\partial\bar{\partial}(\omega_u - \alpha'\|\Omega\|_{\omega_u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu. \quad (2.28)$$

Since we wish to apply techniques of geometric flows, it is useful to re-express the flow entirely in terms of curvature. If we denote by Ric_{ω_u} the Chern-Ricci tensor of ω_u , we have

$$\text{Ric}_{\omega_u} = -2\partial\bar{\partial}u \quad (2.29)$$

since the metric $\hat{\omega}$ is Ricci-flat. Thus the anomaly flow can be rewritten as the following flow of metrics on X ,

$$\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}(\omega - \alpha'\|\Omega\|_\omega \rho) - \frac{\alpha'}{8}\text{Ric}_\omega \wedge \text{Ric}_\omega + \mu \quad (2.30)$$

which we can take now as our starting point. Here we have suppressed the subindex u in ω_u .

A technical issue in anomaly flows is that they are formulated in terms of flows for $\|\Omega\|_\omega \omega^2$, and not of ω itself. This issue was addressed in all generality in [23] in dimension 3. For the above anomaly flow on the surface X arising from the Goldstein-Prokushkin fibration, the metric ω is already characterized by its volume form, and we can proceed more directly as follows.

First, using the two-dimensional identity $2\partial_t\omega \wedge \omega = (\partial_t \log \omega^2)\omega^2$ and the fact that $\omega^2 = \|\Omega\|_\omega^{-2}\hat{\omega}^2$, we can rewrite the left hand side as

$$\partial_t(\|\Omega\|_\omega \omega^2) = \|\Omega\|_\omega(\partial_t \log \|\Omega\|_\omega \omega^2 + 2\partial_t\omega \wedge \omega) = -\|\Omega\|_\omega(\partial_t \log \|\Omega\|_\omega)\omega^2. \quad (2.31)$$

Next, we work out the right hand side more explicitly. Following [23], we define the torsion $T(\omega) = \frac{1}{2}T_{\bar{k}pq} dz^q \wedge dz^p \wedge d\bar{z}^k$ of a Hermitian metric ω by

$$T = i\partial\omega, \quad \bar{T} = -i\bar{\partial}\omega. \quad (2.32)$$

and we also introduce the $(1,0)$ -form T_m and the $(0,1)$ -form \bar{T} by

$$T_m = g^{j\bar{k}}T_{\bar{k}jm}, \quad \bar{T}_{\bar{m}} = g^{j\bar{k}}\bar{T}_{k\bar{j}\bar{m}}. \quad (2.33)$$

Then

$$(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = R_{\bar{k}j\bar{\ell}m} - R_{\bar{k}m\bar{\ell}j} + R_{\bar{\ell}m\bar{k}j} - R_{\bar{\ell}j\bar{k}m} + g^{s\bar{r}} T_{\bar{r}mj} \bar{T}_{s\bar{k}\bar{\ell}}. \quad (2.34)$$

In general, there are several notions of Ricci curvature for Hermitian metrics, given by

$$R_{\bar{k}j} = R_{\bar{k}j}^p{}_p, \quad \tilde{R}_{\bar{k}j} = R^p{}_{p\bar{k}j}, \quad R'_{\bar{k}j} = R_{\bar{k}}^p{}_{pj}, \quad R''_{\bar{k}j} = R^p{}_{j\bar{k}p} \quad (2.35)$$

For metrics of the form $\omega = e^u \hat{\omega}$, where $\hat{\omega}$ is Kähler and Ricci-flat, the following important relations between torsion and curvature hold,

$$T_q(\omega) = \partial_q \log \|\Omega\|_\omega, \quad \bar{T}_{\bar{q}}(\omega) = \partial_{\bar{q}} \log \|\Omega\|_\omega \quad (2.36)$$

and

$$\begin{aligned} R_{\bar{k}j}(\omega) &= 2\nabla_{\bar{k}}T_j(\omega) = 2\nabla_j\bar{T}_{\bar{k}}(\omega), \\ R'_{\bar{k}j}(\omega) &= R''_{\bar{k}j}(\omega) = \frac{1}{2}R_{\bar{k}j}(\omega). \end{aligned} \quad (2.37)$$

Also, because $\hat{\omega}$ is Kähler, we have

$$T(\omega) = i\partial u \wedge \omega \quad (2.38)$$

so that the $(1,0)$ -form T_m actually determines in our case the full $(2,1)$ torsion tensor $T(\omega)$. Henceforth, unless explicitly indicated otherwise, we shall designate by T the $(1,0)$ -form $T_m dz^m$ rather than the $(2,1)$ -form $i\partial\omega$. For example, the norm $|T|^2$ will designate the expression

$$|T|^2 = g^{m\bar{\ell}} T_m \bar{T}_{\bar{\ell}} \quad (2.39)$$

rather than $|i\partial\omega|^2$ (which can be verified to be equal to $2|T|^2$).

Using these relations, and the fact that we are in dimension 2, we find

$$i\partial\bar{\partial}\omega = \frac{1}{2}(-R + 2|T|^2)\frac{\omega^2}{2}. \quad (2.40)$$

Substituting this equation and (2.31) in the flow (2.30), we obtain

$$\partial_t \log \|\Omega\|_\omega = \frac{1}{\|\Omega\|_\omega} \left(\frac{R}{2} - |T|^2 + 2\alpha' \frac{i\partial\bar{\partial}(\|\Omega\|_\omega \rho)}{\omega^2} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - \|\Omega\|_\omega^2 \tilde{\mu} \right), \quad (2.41)$$

where we have introduced the time-independent, scalar function $\tilde{\mu}$ by $\mu = \tilde{\mu} \frac{\omega^2}{2}$, and the σ_2 operator with respect to the evolving metric

$$2\sigma_2(i\text{Ric}_\omega) \frac{\omega^2}{2} = i\text{Ric}_\omega \wedge i\text{Ric}_\omega. \quad (2.42)$$

Since the metric $\omega = e^u \hat{\omega}$ is entirely determined by the conformal factor e^u , this flow for the volume form is equivalent to the flow of metrics (1.1) quoted in the Introduction. The flow in terms of the conformal factor u is easily worked out to be given by the equation (1.4).

2.4 Comparisons between the 3-dimensional anomaly flow and its 2-dimensional reduction

It may be noteworthy that the flow (2.30) retains many of the features of the original anomaly flow in 3-dimensions. Indeed, as shown in [23], the conformally balanced condition (2.5) in dimension 3 implies that the Hermitian metric χ on the 3-fold Y satisfies exactly the same relations (2.36) and (2.3) between torsion and curvature as the metric $\omega = e^u \hat{\omega}$ on the surface X ,

$$T_q(\chi) = \partial_q \log \|\Omega_Y\|_\chi, \quad \bar{T}_{\bar{q}}(\chi) = \partial_{\bar{q}} \log \|\Omega_Y\|_\chi \quad (2.43)$$

and

$$\begin{aligned} R_{\bar{k}j}(\chi) &= 2\nabla_{\bar{k}}T_j(\chi) = 2\nabla_j\bar{T}_{\bar{k}}(\chi), \\ R'_{\bar{k}j}(\chi) &= R''_{\bar{k}j}(\chi) = \frac{1}{2}R_{\bar{k}j}(\chi). \end{aligned} \tag{2.44}$$

This suggests that the flow (1.1) is interesting not just as a special case of the general anomaly flow, but also as a good model for developing general methods for studying the flow.

2.5 Starting the Flow

In [22] general conditions were given for the short-time existence of the anomaly flow, using the Nash-Moser implicit function theorem. However, the short-time existence of the flow can be seen more directly from the parabolicity of the flow, which holds when the form

$$\omega' = e^u\hat{\omega} + \alpha'e^{-u}\rho + \alpha'i\partial\bar{\partial}u > 0, \tag{2.45}$$

is positive definite. This can be seen from the scalar equation (1.4). We will always assume that we start the flow from a large constant multiple of the background metric

$$u(x, 0) = \log M \gg 1, \quad \omega(0) = e^{u(0)}\hat{\omega} = M\hat{\omega}. \tag{2.46}$$

Recall that μ is defined in (2.24). In all that follows, we will assume that the cohomological condition

$$\int_X \mu = 0, \tag{2.47}$$

is satisfied. Integrating (2.28) and using the fact that $\|\Omega\|_{\omega}\omega^2 = e^u\hat{\omega}^2$ gives the following conservation law

$$\frac{\partial}{\partial t} \int_X e^u \frac{\hat{\omega}^2}{2!} = 0. \tag{2.48}$$

Hence

$$\int_X e^u \frac{\hat{\omega}^2}{2!} = M, \tag{2.49}$$

along the flow.

3 The C^0 estimate of the conformal factor

In this section, we will work with equation (2.28), since it will be easier to work with differential forms to obtain integral estimates. We let $\hat{\omega}$ denote the fixed background Kähler form of X . We can rescale $\hat{\omega}$ such that $\int_X \frac{\hat{\omega}^2}{2!} = 1$. We will omit the background volume

form $\frac{\hat{\omega}^2}{2!}$ when integrating scalar functions. The starting point for the Moser iteration argument is to compute the quantity

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega', \quad (3.1)$$

in two different ways. Recall that ω' is defined in (2.45). On one hand, by the definition of ω' and Stokes' theorem, we have

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' = \int_X \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}(e^{-ku}). \quad (3.2)$$

Expanding

$$\begin{aligned} \int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' &= k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u \\ &\quad - k \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial} u. \end{aligned} \quad (3.3)$$

On the other hand, without using Stokes' theorem, we obtain

$$\begin{aligned} \int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' &= k^2 \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' - k \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial} u \\ &\quad - \alpha' k \int_X e^{-ku} i\partial\bar{\partial} u \wedge i\partial\bar{\partial} u. \end{aligned} \quad (3.4)$$

We equate (3.3) and (3.4)

$$\begin{aligned} 0 &= -k^2 \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' + k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u \\ &\quad + \alpha' k \int_X e^{-ku} i\partial\bar{\partial} u \wedge i\partial\bar{\partial} u. \end{aligned} \quad (3.5)$$

Using equation (2.28) and that $\|\Omega\|_\omega \omega^2 = e^u \hat{\omega}^2$,

$$\begin{aligned} 0 &= -k^2 \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' + k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u \\ &\quad - 2k \int_X e^{-ku} \mu - 2k \int_X e^{-ku} i\partial\bar{\partial}(e^u \hat{\omega} - \alpha' e^{-u} \rho) + 4k \int_X e^{-(k-1)u} \partial_t u \frac{\hat{\omega}^2}{2!}. \end{aligned} \quad (3.6)$$

Expanding out terms and dividing by $2k$ yields

$$\begin{aligned} 0 &= -\frac{k}{2} \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' + \frac{k}{2} \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u - \int_X e^{-ku} \mu \\ &\quad - \int_X e^{-(k-1)u} i\partial\bar{\partial} u \wedge \hat{\omega} - \int_X e^{-(k-1)u} i\partial u \wedge \bar{\partial} u \wedge \hat{\omega} - \alpha' \int_X e^{-(k+1)u} i\partial\bar{\partial} u \wedge \rho \\ &\quad + \alpha' \int_X e^{-(k+1)u} i\partial u \wedge \bar{\partial} u \wedge \rho + \alpha' \int_X e^{-(k+1)u} i\partial\bar{\partial} \rho \\ &\quad - 2\alpha' \text{Re} \int_X e^{-(k+1)u} i\partial u \wedge \bar{\partial} \rho + 2 \int_X e^{-(k-1)u} \partial_t u \frac{\hat{\omega}^2}{2!}. \end{aligned} \quad (3.7)$$

Integration by parts gives

$$\begin{aligned}
0 &= -\frac{k}{2} \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' - \frac{k}{2} \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u \\
&\quad - \int_X e^{-ku} \mu + \alpha' \int_X e^{-(k+1)u} i\partial \bar{\partial} \rho - \alpha' \operatorname{Re} \int_X e^{-(k+1)u} i\partial u \wedge \bar{\partial} \rho \\
&\quad + 2 \int_X e^{-(k-1)u} \partial_t u \frac{\hat{\omega}^2}{2!}.
\end{aligned} \tag{3.8}$$

One more integration by parts yields the following identity:

$$\begin{aligned}
&\frac{k}{2} \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u + \frac{\partial}{\partial t} \frac{2}{k-1} \int_X e^{-(k-1)u} \frac{\hat{\omega}^2}{2!} \\
&= -\frac{k}{2} \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' - \int_X e^{-ku} \mu + \left(\alpha' - \frac{\alpha'}{k+1}\right) \int_X e^{-(k+1)u} i\partial \bar{\partial} \rho.
\end{aligned} \tag{3.9}$$

The identity (3.9) will be useful later to control the infimum of u , but to control the supremum of u , we replace k with $-k$ in (3.9). Then, for $k \neq 1$,

$$\begin{aligned}
&\frac{k}{2} \int_X e^{(k+1)u} \{\hat{\omega} + \alpha' e^{-2u} \rho\} \wedge i\partial u \wedge \bar{\partial} u + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \frac{\hat{\omega}^2}{2!} \\
&= -\frac{k}{2} \int_X e^{ku} i\partial u \wedge \bar{\partial} u \wedge \omega' + \int_X e^{ku} \mu - \left(\alpha' - \frac{\alpha'}{1-k}\right) \int_X e^{(k-1)u} i\partial \bar{\partial} \rho.
\end{aligned} \tag{3.10}$$

3.1 Estimating the supremum

Proposition 1 *Suppose the flow exists for $t \in [0, T] \subset [0, 1]$, and that $\inf_X e^u \geq 1$ and $\alpha' e^{-2u} \rho \geq -\frac{1}{2} \hat{\omega}$ for all time $t \in [0, T]$. Then*

$$\sup_{X \times [0, T]} e^u \leq C_1 M, \tag{3.11}$$

where C_1 only depends on $(X, \hat{\omega})$, ρ , μ , α' .

Proof: As long as the flow exists, we have

$$i\partial u \wedge \bar{\partial} u \wedge \omega' \geq 0. \tag{3.12}$$

Let $\beta = \frac{n}{n-1} = 2$. We can use (3.12), (3.10), and $\alpha e^{-2u} \rho \geq -\frac{1}{2} \hat{\omega}$ to derive the following estimate for any $k \geq \beta$

$$\begin{aligned}
&\frac{k}{4} \int_X e^{(k+1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \\
&\leq (\|\mu\|_{L^\infty} + 2|\alpha'| \|\rho\|_{C^2}) \left(\int_X e^{ku} + \int_X e^{(k-1)u} \right).
\end{aligned} \tag{3.13}$$

Here we omit the background volume form $\frac{\hat{\omega}^2}{2!}$ when integrating scalars.

Let $0 < \tau < \tau' < T$. Let $\zeta(t) \geq 0$ be a monotone function which is zero for $t \leq \tau$ and identically 1 for $t \geq \tau'$. Multiplying the previous inequality by ζ gives, for any $k \geq \beta$,

$$\begin{aligned} & \frac{k\zeta}{4} \int_X e^{(k+1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2\zeta}{k+1} \int_X e^{(k+1)u} \\ & \leq (\|\mu\|_{L^\infty} + 2\|\alpha'\| \|\rho\|_{C^2}) \left\{ \zeta \int_X e^{(k-1)u} + \zeta \int_X e^{ku} \right\} + \frac{2\zeta'}{k+1} \int_X e^{(k+1)u}. \end{aligned} \quad (3.14)$$

Let $\tau' < s \leq T$. Integrating from τ to s yields

$$\frac{k}{4} \int_{\tau'}^s \int_X e^{(k+1)u} |Du|^2 + \frac{2}{k+1} \int_X e^{(k+1)u}(s) \quad (3.15)$$

$$\leq C \left\{ \int_\tau^T \int_X e^{(k-1)u} + \int_\tau^T \int_X e^{ku} + \frac{1}{\tau' - \tau} \int_\tau^T \int_X e^{(k+1)u} \right\}, \quad (3.16)$$

for any $k \geq \beta$, where C only depends on α' , ρ , μ . We rearrange this inequality to obtain, for $k \geq \beta + 1$,

$$\begin{aligned} & \frac{(k-1)}{k} \int_{\tau'}^s \int_X |De^{\frac{k}{2}u}|^2 + \int_X e^{ku}(s) \\ & \leq Ck \left\{ \int_\tau^T \int_X e^{(k-2)u} + \int_\tau^T \int_X e^{(k-1)u} + \frac{1}{\tau' - \tau} \int_\tau^T \int_X e^{ku} \right\}. \end{aligned} \quad (3.17)$$

Using $e^{-u} \leq 1$,

$$\int_{\tau'}^s \int_X |De^{\frac{k}{2}u}|^2 + \int_X e^{ku}(s) \leq Ck \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \left\{ \int_\tau^T \int_X e^{ku} \right\}. \quad (3.18)$$

The Sobolev inequality gives us

$$\left(\int_X e^{k\beta u} \right)^{\frac{1}{\beta}} \leq C'_X \left(\int_X |e^{\frac{k}{2}u}|^2 + \int_X |De^{\frac{k}{2}u}|^2 \right). \quad (3.19)$$

where C'_X is the Sobolev constant on manifold $(X, \hat{\omega})$. Let β^* be such that $\frac{1}{\beta} + \frac{1}{\beta^*} = 1$. By Hölder's inequality and the Sobolev inequality,

$$\begin{aligned} \int_{\tau'}^T \int_X e^{ku} e^{\frac{k}{\beta^*}u} & \leq \int_{\tau'}^T \left(\int_X e^{k\beta u} \right)^{1/\beta} \left(\int_X e^{ku} \right)^{1/\beta^*} \\ & \leq C'_X \sup_{t \in [\tau', T]} \left(\int_X e^{ku} \right)^{1/\beta^*} \int_{\tau'}^T \left\{ \int_X e^{ku} + \int_X |De^{\frac{k}{2}u}|^2 \right\}. \end{aligned} \quad (3.20)$$

Using estimate (3.18), and defining $\gamma = 1 + \frac{1}{\beta^*} = 1 + \frac{1}{2}$, we have for $k \geq 1 + \beta$,

$$\left(\int_{\tau'}^T \int_X e^{\gamma ku} \right)^{1/\gamma} \leq Ck \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \int_\tau^T \int_X e^{ku}. \quad (3.21)$$

Choose $0 < T_0 \leq T \leq 1$. We will iterate with $\tau_k = (1 - \gamma^{-(k+1)})T_0$. Note $\tau_k \geq \frac{T_0}{3}$.

$$\left(\int_{\tau_{k+1}}^T \int_X e^{\gamma^{k+1}u} \right)^{1/\gamma^{k+1}} \leq \left\{ C\gamma^k + \frac{CT_0\gamma^{-1}}{1 - \gamma^{-1}} \right\}^{1/\gamma^k} \left\{ \int_{\tau_k}^T \int_X e^{\gamma^k u} \right\}^{1/\gamma^k}. \quad (3.22)$$

Iterating, we see that for $p = \gamma^{\kappa_0} \geq 1 + \beta$, there holds

$$\sup_{X \times [T_0, T]} e^{u(z, t)} \leq C \|e^u\|_{L^p(X \times [\frac{T_0}{3}, T])}, \quad (3.23)$$

where C only depends on $(X, \hat{\omega})$, ρ , μ , and α' . To relate the L^p norm of e^u to $\int_X e^u = M$, we can use a standard scaling argument. By the previous estimate

$$\sup_{X \times [0, T]} e^u \leq C \left(\int_{X \times [0, T]} e^u e^{pu-u} \right)^{1/p} \leq C \left(\sup_{X \times [0, T]} e^{(p-1)u} \right)^{\frac{1}{p}} \left(\int_{X \times [0, T]} e^u \right)^{1/p}. \quad (3.24)$$

Therefore

$$\sup_{X \times [0, T]} e^u \leq C_1 M, \quad (3.25)$$

for any $T \leq 1$.

3.2 Estimating the infimum

We introduce the constant

$$\theta = \frac{1}{2C_1 - 1}. \quad (3.26)$$

Note that since $C_1 \geq 1$, we must have $0 < \theta \leq 1$. Fix a small constant $0 < \delta < 1$ such that

$$\delta < \frac{\theta}{4C_X(|\alpha'| \|\rho\|_{C^2} + \|\mu\|_{C^0})}, \quad \text{and} \quad \alpha' \delta^2 \rho \geq -\frac{1}{2} \hat{\omega}, \quad (3.27)$$

where C_X is the Poincaré constant for the reference Kähler manifold $(X, \hat{\omega})$. Define

$$S_\delta := \{t \in [0, T] \subset [0, 1] : \sup_X e^{-u} \leq \delta\}. \quad (3.28)$$

Recall that we start the flow at $u_0 = \log M$. It follows that if $M > \delta^{-1}$, then the flow starts in the region S_δ . At any time $\hat{t} \in S_\delta$, we consider $U = \{z \in X : e^{-u} \leq \frac{2}{M}\}$. Then by (3.25),

$$M = \int_U e^u + \int_{X \setminus U} e^u \leq |U| \sup_X e^u + (1 - |U|) \frac{M}{2} \leq C_1 M |U| + (1 - |U|) \frac{M}{2}. \quad (3.29)$$

It follows that at any \hat{t} ,

$$|U| > \theta > 0. \quad (3.30)$$

We will also need the constant $C_0 > 1$ defined by

$$C_0 = \frac{1}{1 - \frac{\theta}{4}} \left(1 + \frac{2}{\theta} \right) \left(\frac{2}{\theta^2} \right). \quad (3.31)$$

3.2.1 Integral estimate

Proposition 2 *For all $t \in S_\delta$, there holds*

$$\int_X e^{-u} \leq \frac{2C_0}{M}. \quad (3.32)$$

Proof: Recall that we start the flow at $u_0 = \log M$. At $t = 0$, we have $\int_X e^{-u} = \frac{1}{M} < \frac{2C_0}{M}$. Suppose $\hat{t} \in S_\delta$ is the first time when we reach $\int_X e^{-u} = \frac{2C_0}{M}$. Then we must have

$$\frac{\partial}{\partial t} \Big|_{t=\hat{t}} \int_X e^{-u} \geq 0. \quad (3.33)$$

Setting $k = 2$ in (3.9) and dropping the negative term involving $\omega' \geq 0$, we have

$$\begin{aligned} & \int_X e^{-u} \{ \hat{\omega} + \alpha' e^{-2u} \rho \} \wedge i \partial u \wedge i \bar{\partial} u + 2 \frac{\partial}{\partial t} \int_X e^{-u} \\ & \leq \left(|\alpha'| \|\rho\|_{C^2} \int_X e^{-3u} + \|\mu\|_{C^0} \int_X e^{-2u} \right). \end{aligned} \quad (3.34)$$

Since $\frac{\partial}{\partial t} \int_X e^{-u} \geq 0$, and $e^{-u} \leq \delta < 1$, there holds at \hat{t} ,

$$\int_X |D e^{-\frac{u}{2}}|^2 \leq (|\alpha'| \|\rho\|_{C^2} + \|\mu\|_{C^0}) \delta \int_X e^{-u}. \quad (3.35)$$

By the Poincaré inequality

$$\int_X e^{-u} - \left(\int_X e^{-\frac{u}{2}} \right)^2 = \int_X \left| e^{-\frac{u}{2}} - \int_X e^{-\frac{u}{2}} \right|^2 \leq C_X \int_X |D e^{-\frac{u}{2}}|^2. \quad (3.36)$$

By (3.27), we have

$$\int_X e^{-u} - \left(\int_X e^{-\frac{u}{2}} \right)^2 \leq \frac{\theta}{4} \int_X e^{-u}, \quad (3.37)$$

and it implies

$$\int_X e^{-u} \leq \frac{1}{1 - \frac{\theta}{4}} \left(\int_X e^{-\frac{u}{2}} \right)^2. \quad (3.38)$$

Let $\varepsilon > 0$. We may use the measure estimate and (3.38) to obtain

$$\begin{aligned} \left(\int_X e^{-\frac{u}{2}} \right)^2 & \leq \left(1 + \frac{2}{\theta} \right) \left(\int_U e^{-\frac{u}{2}} \right)^2 + \left(1 + \frac{\theta}{2} \right) \left(\int_{X \setminus U} e^{-\frac{u}{2}} \right)^2 \\ & \leq \left(1 + \frac{2}{\theta} \right) |U| \int_U e^{-u} + \left(1 + \frac{\theta}{2} \right) (1 - |U|) \int_{X \setminus U} e^{-u} \\ & \leq \left(1 + \frac{2}{\theta} \right) \frac{2}{M} + \left(1 + \frac{\theta}{2} \right) (1 - \theta) \frac{1}{1 - \frac{\theta}{4}} \left(\int_X e^{-\frac{u}{2}} \right)^2. \end{aligned} \quad (3.39)$$

Thus

$$\left(\int_X e^{-\frac{u}{2}}\right)^2 \leq \left(1 + \frac{2}{\theta}\right) \frac{2}{M} \left(\frac{1}{1 - (1 + \frac{\theta}{2})(1 - \theta)(1 - \frac{\theta}{4})^{-1}}\right). \quad (3.40)$$

Since $0 < \theta \leq 1$,

$$\left(1 + \frac{\theta}{2}\right)(1 - \theta)\left(1 - \frac{\theta}{4}\right)^{-1} \leq 1 + \theta^2. \quad (3.41)$$

Using this and (3.38),

$$\int_X e^{-u} \leq \frac{1}{1 - \frac{\theta}{4}} \left(1 + \frac{2}{\theta}\right) \left(\frac{2}{\theta^2}\right) \frac{1}{M} = \frac{C_0}{M}. \quad (3.42)$$

This contradicts that $\int_X e^{-u} = \frac{2C_0}{M}$ at t_0 . It follows that $\int_X e^{-u}$ stays less than $\frac{2C_0}{M}$ for all time $t \in S_\delta$.

3.2.2 Iteration

Proposition 3 *For all $t \in S_\delta$, there holds*

$$\left(\sup_X e^{-u}\right)(t) \leq \frac{C_2}{M}, \quad (3.43)$$

where C_2 only depends on $(X, \hat{\omega})$, ρ , μ , α' .

Proof: We can drop the negative terms involving $\omega' \geq 0$ and use $\alpha' e^{-2u} \rho \geq -\frac{1}{2}\omega$ in (3.9) to obtain the estimate

$$\frac{k}{2} \int_X e^{-(k-1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2}{k-1} \int_X e^{-(k-1)u} \leq C \left(\int_X e^{-(k+1)u} + \int_X e^{-ku} \right). \quad (3.44)$$

Let $0 < \tau < \tau' < T$, where T is such that $[0, T) \subset S_\delta$. Let $\zeta(t) \geq 0$ be a monotone function which is zero for $t \leq \tau$ and identically 1 for $t \geq \tau'$. Multiplying the previous inequality by ζ gives

$$\frac{k\zeta}{2} \int_X e^{-(k-1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2\zeta}{k-1} \int_X e^{-(k-1)u} \leq C \left\{ \zeta \int_X e^{-(k+1)u} + \zeta \int_X e^{-ku} + \zeta' \int_X e^{-(k-1)u} \right\}. \quad (3.45)$$

Let $\tau' < s \leq T$. Integrating from τ to s

$$\begin{aligned} & \frac{k}{2} \int_{\tau'}^s \int_X e^{-(k-1)u} |Du|^2 + \frac{2}{k-1} \int_X e^{-(k-1)u}(s) \\ & \leq C \left\{ \int_{\tau}^T \int_X e^{-(k+1)u} + \int_{\tau}^T \int_X e^{-ku} + \frac{1}{\tau' - \tau} \int_{\tau}^T \int_X e^{-(k-1)u} \right\}. \end{aligned} \quad (3.46)$$

We rearrange this inequality to obtain, for $k > 0$,

$$\begin{aligned} & \int_{\tau'}^s \int_X |De^{-\frac{k}{2}u}|^2 + 2 \int_X e^{-ku}(s) \\ & \leq Ck \left\{ \int_{\tau}^T \int_X e^{-(k+2)u} + \int_{\tau}^T \int_X e^{-(k+1)u} + \frac{1}{\tau' - \tau} \int_{\tau}^T \int_X e^{-ku} \right\}. \end{aligned} \quad (3.47)$$

By $e^{-u} \leq \delta < 1$, we have

$$\int_{\tau'}^s \int_X |De^{-\frac{k}{2}u}|^2 + 2 \int_X e^{-ku}(s) \leq Ck \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \left\{ \int_{\tau}^T \int_X e^{-ku} \right\}. \quad (3.48)$$

Recall that we denote $\beta = \frac{n}{n-1} = 2$, β^* such that $\frac{1}{\beta} + \frac{1}{\beta^*} = 1$, and $\gamma = 1 + \frac{1}{\beta^*}$. By the Sobolev inequality

$$\begin{aligned} \int_{\tau'}^T \int_X e^{-ku} e^{-\frac{k}{\beta^*}u} &\leq \int_{\tau'}^T \left(\int_X e^{-k\beta u} \right)^{1/\beta} \left(\int_X e^{-ku} \right)^{1/\beta^*} \\ &\leq C \sup_{t \in [\tau', T]} \left(\int_X e^{-ku} \right)^{1/\beta^*} \int_{\tau'}^T \left\{ \int_X e^{-ku} + \int_X |De^{-\frac{k}{2}u}|^2 \right\}. \end{aligned} \quad (3.49)$$

Using estimate (3.48), we arrive at

$$\left(\int_{\tau'}^T \int_X e^{-\gamma ku} \right)^{1/\gamma} \leq Ck \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \left\{ \int_{\tau}^T \int_X e^{-ku} \right\}. \quad (3.50)$$

Choose $0 < T_0 \leq T \leq 1$. Iterating with $\tau_k = (1 - \gamma^{-(k+1)})T_0$, we see that for any $t \in [T_0, T]$ we have the C^0 estimate

$$e^{-u(z,t)} \leq C \|e^{-u}\|_{L^1(S \times [\frac{T_0}{3}, T])}. \quad (3.51)$$

By Proposition 2,

$$\sup_{X \times [0, T]} e^{-u(z,t)} \leq \frac{C_2}{M}. \quad (3.52)$$

Theorem 3 *Suppose the flow exists for $t \in [0, T]$, and initially starts with $u_0 = \log M$. There exists $M_0 \gg 1$ such that for all $M \geq M_0$, there holds*

$$\sup_{X \times [0, T]} e^u \leq C_1 M, \quad \sup_{X \times [0, T]} e^{-u} \leq \frac{C_2}{M}, \quad (3.53)$$

where C_2, C_1 only depends on $(X, \hat{\omega})$, ρ , μ , α' .

Proof: By Propositions 1 and 3, the estimates hold as long as we stay in S_δ . Choose M_0 such that

$$\frac{C_2}{M_0} < \delta, \quad (3.54)$$

where recall δ is defined in (3.27). Then at $t = 0$, we have $e^{-u_0} < \delta$, and the estimate is preserved on $[0, 1]$. It follows that the theorem holds for $T \leq 1$.

Suppose the theorem holds for $T \leq n$. For $n \leq T \leq n+1$, consider $\hat{u}(t) = u(t+n)$. Then at $t = 0$, $e^{-\hat{u}(0)} < \delta$, and the estimate $e^{-\hat{u}(t)} \leq \frac{C_2}{M} < \delta$ holds on $[0, 1]$. Hence the theorem holds for arbitrary T . Q.E.D.

4 Evolution of the torsion

Before proceeding, we clearly state the conventions and notation that will be used for the maximum principle estimates of Sections §4-6. All norms from this point on will be with respect to the evolving metric $\omega = e^u \hat{\omega}$, unless denoted otherwise. We will write $\omega = ig_{\bar{k}j} dz^j \wedge d\bar{z}^k$. We will use the Chern connection of ω to differentiate

$$\nabla_{\bar{k}} V^\alpha = \partial_{\bar{k}} V^\alpha, \quad \nabla_k V^\alpha = g^{\alpha\bar{\beta}} \partial_k (g_{\bar{\beta}\gamma} V^\gamma). \quad (4.1)$$

The curvature of the metric ω is

$$R_{\bar{k}j}{}^\alpha{}_\beta = -\partial_{\bar{k}}(g^{\alpha\bar{\gamma}} \partial_j g_{\bar{\gamma}\beta}) = \hat{R}_{\bar{k}j}{}^\alpha{}_\beta - u_{\bar{k}j} \delta^\alpha{}_\beta. \quad (4.2)$$

The torsion tensor of the metric ω is $T_{\bar{k}mj} = \partial_m g_{\bar{k}j} - \partial_j g_{\bar{k}m}$, and since $\hat{\omega}$ has zero torsion, we may compute

$$T^\lambda{}_{mj} = g^{\lambda\bar{k}} T_{\bar{k}mj} = u_m \delta^\lambda{}_j - u_j \delta^\lambda{}_m. \quad (4.3)$$

We note the following formulas for the torsion and Chern-Ricci curvature of the evolving metric

$$R_{\bar{k}j} = R_{\bar{k}j}{}^\alpha{}_\alpha = -2u_{\bar{k}j}, \quad T_j = T^\lambda{}_{\lambda j} = -\partial_j u. \quad (4.4)$$

Recall that $|T|^2$ refers to the norm of T_j , as noted in (2.39). We will often use the following commutation formulas to exchange covariant derivatives

$$[\nabla_j, \nabla_{\bar{k}}] V_i = -R_{\bar{k}j}{}^p{}_i V_p, \quad [\nabla_j, \nabla_k] V_i = -T^\lambda{}_{jk} \nabla_\lambda V_i. \quad (4.5)$$

To handle the differentiation of the equation, we will rewrite the terms involving ρ in the flow (1.1). Compute

$$\begin{aligned} -\alpha' i \partial \bar{\partial} (e^{-u} \rho) &= -\alpha' e^{-u} i \partial \bar{\partial} \rho + 2\alpha' \text{Re}\{e^{-u} i \partial u \wedge \bar{\partial} \rho\} \\ &\quad + \alpha' e^{-u} i \partial \bar{\partial} u \wedge \rho - \alpha' i e^{-u} \partial u \wedge \bar{\partial} u \wedge \rho. \end{aligned} \quad (4.6)$$

We introduce the notation

$$-\alpha' i \partial \bar{\partial} (e^{-u} \rho) = \left(-\alpha' e^{-u} \psi_\rho + \alpha' e^{-u} \text{Re}\{b_\rho^i u_i\} + \alpha' e^{-u} \tilde{\rho}^{j\bar{k}} u_{\bar{k}j} - \alpha' e^{-u} \tilde{\rho}^{p\bar{q}} u_p \bar{u}_{\bar{q}} \right) \frac{\hat{\omega}^2}{2}, \quad (4.7)$$

where $\psi_\rho(z)$, $b_\rho^i(z)$, $\tilde{\rho}^{j\bar{k}}(z)$ are defined one by one corresponding to the previous expression. We note that ψ_ρ , b_ρ^i , $\tilde{\rho}^{j\bar{k}}$ are bounded in C^∞ by constants depending only on the form ρ and the background metric $\hat{\omega}$. We also note that $\tilde{\rho}^{j\bar{k}}$ is Hermitian since ρ is real. We may rewrite this expression as

$$-\alpha' i \partial \bar{\partial} (e^{-u} \rho) = \left(-\alpha' e^{-3u} \psi_\rho - \alpha' e^{-3u} \text{Re}\{b_\rho^i T_i\} - \frac{\alpha'}{2} e^{-3u} \tilde{\rho}^{j\bar{k}} R_{\bar{k}j} - \alpha' e^{-3u} \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} \right) \frac{\omega^2}{2}. \quad (4.8)$$

With all the introduced notation, we can write the flow (1.1) in the following way.

$$\partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega} \left(-\frac{R}{2} - \frac{\alpha'}{2} \|\Omega\|_\omega^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) + |T|^2 + \|\Omega\|_\omega^2 \nu \right) g_{\bar{k}j}, \quad (4.9)$$

where

$$\nu = -\alpha' \|\Omega\|_\omega \psi_\rho - \alpha' \|\Omega\|_\omega \text{Re}\{b_\rho^i T_i\} - \alpha' \|\Omega\|_\omega \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} + \tilde{\mu}. \quad (4.10)$$

In the following, we will use $\|\Omega\|$ to replace $\|\Omega\|_\omega$ for simplicity, if there is no confusing of the notation.

4.1 Torsion tensor

Using $\|\Omega\| = e^{-u}$ and $g_{\bar{k}j} = e^u \hat{g}_{\bar{k}j}$, (4.9) implies the following evolution of $\|\Omega\|$,

$$\partial_t \log \|\Omega\| = \frac{1}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - |T|^2 - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - \|\Omega\|^2 \nu \right). \quad (4.11)$$

Using (2.36) and (4.11), we evolve

$$\begin{aligned} \partial_t T_j &= \partial_j \partial_t \log \|\Omega\| \\ &= \nabla_j \left\{ \frac{1}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - |T|^2 - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - \|\Omega\|^2 \nu \right) \right\}. \end{aligned} \quad (4.12)$$

Using $\partial_j \|\Omega\| = \|\Omega\| T_j$ and the definition of ν (4.10), a straightforward computation gives

$$\begin{aligned} \partial_t T_j &= \frac{1}{2\|\Omega\|} \left\{ -\frac{1}{2} T_j R + T_j |T|^2 + \frac{\alpha'}{4} T_j \sigma_2(i\text{Ric}_\omega) \right. \\ &\quad \left. + \frac{1}{2} \nabla_j R + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j R_{\bar{q}p} - \nabla_j |T|^2 - \frac{\alpha'}{4} \nabla_j \sigma_2(i\text{Ric}_\omega) + E_j \right\}, \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} E_j &= 2\alpha' \|\Omega\|^3 \psi_\rho T_j + 2\alpha' \|\Omega\|^3 \text{Re}\{b_\rho^i T_i\} T_j + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} T_j \\ &\quad + 2\alpha' \|\Omega\|^3 (\tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}}) T_j - \|\Omega\|^2 \tilde{\mu} T_j + \alpha' \|\Omega\|^3 \nabla_j \psi_\rho \\ &\quad + \alpha' \|\Omega\|^3 \text{Re}\{\nabla_j b_\rho^i T_i\} + \alpha' \|\Omega\|^3 \text{Re}\{b_\rho^i \nabla_j T_i\} + \frac{\alpha'}{2} \|\Omega\|^3 (\nabla_j \tilde{\rho}^{p\bar{q}}) R_{\bar{q}p} \\ &\quad + \alpha' \|\Omega\|^3 (\nabla_j \tilde{\rho}^{p\bar{q}}) T_p \bar{T}_{\bar{q}} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j T_p \bar{T}_{\bar{q}} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p R_{\bar{q}j} \\ &\quad - \|\Omega\|^2 \nabla_j \tilde{\mu}. \end{aligned} \quad (4.14)$$

Our reason for treating E_j as an error term is that the C^0 estimate tells us that $\|\Omega\| = e^{-u} \ll 1$ if we start the flow from a large enough constant $\log M$. As we will see, the terms appearing in E_j will only slightly perturb the coefficients of the leading terms in the proof of Theorem 4.

We need to express the highest order terms in (4.13) as the linearized operator acting on torsion. First, we write the Ricci curvature in terms of the conformal factor

$$\nabla_j R_{\bar{q}p} = -2\nabla_j \nabla_p \nabla_{\bar{q}} u. \quad (4.15)$$

Exchanging covariant derivatives

$$-2\nabla_j \nabla_p \nabla_{\bar{q}} u = -2\nabla_p \nabla_{\bar{q}} \nabla_j u - 2T^\lambda_{pj} \nabla_\lambda \nabla_{\bar{q}} u. \quad (4.16)$$

It follows from (4.4) that

$$\nabla_j R_{\bar{q}p} = 2\nabla_p \nabla_{\bar{q}} T_j + T^\lambda_{pj} R_{\bar{q}\lambda}. \quad (4.17)$$

Hence

$$\begin{aligned} & \nabla_j R - \frac{\alpha'}{2} \nabla_j \sigma_2(i\text{Ric}_\omega) + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j R_{\bar{q}p} \\ &= g^{p\bar{q}} \nabla_j R_{\bar{q}p} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j R_{\bar{q}p} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}} \nabla_j R_{\bar{q}p} \\ &= 2F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j + F^{p\bar{q}} T^\lambda_{pj} R_{\bar{q}\lambda}, \end{aligned} \quad (4.18)$$

where

$$F^{p\bar{q}} := g^{p\bar{q}} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}}. \quad (4.19)$$

The tensor $F^{p\bar{q}}$ is Hermitian, and in Section §5 we will show that $F^{p\bar{q}}$ stays close to $g^{p\bar{q}}$ along the flow. Substituting (4.18) into (4.13)

$$\begin{aligned} \partial_t T_j &= \frac{1}{2\|\Omega\|} \left\{ F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j - \nabla_j |T|^2 - \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2(i\text{Ric}_\omega) \right. \\ &\quad \left. + \frac{1}{2} F^{p\bar{q}} T^\lambda_{pj} R_{\bar{q}\lambda} + T_j |T|^2 + E_j \right\}. \end{aligned} \quad (4.20)$$

Before proceeding, let us define $\sigma_2^{p\bar{q}}$ and rewrite $F^{p\bar{q}}$ in a more familiar form using convenient coordinates. Suppose we work at a point where the evolving metric $g_{i\bar{j}} = \delta_{ij}$ and $R_{\bar{k}j}$ is diagonal. Let $A^i_j = g^{i\bar{k}} R_{\bar{k}j}$. The function $\sigma_2(A^i_j)$ maps a Hermitian endomorphism to the second elementary symmetric polynomial of its eigenvalues. We are working in dimension $n = 2$, so $\sigma_2(A^i_j)$ is the product of the two eigenvalues of A . Our operator $\sigma_2(i\text{Ric}_\omega)$ defined in (2.42) is with respect to the evolving metric ω , so denoting $A^i_j = g^{i\bar{k}} R_{\bar{k}j}$, we have $\sigma_2(i\text{Ric}_\omega) = \sigma_2(A)$. We define $\sigma_2^{p\bar{q}} = \frac{\partial \sigma_2}{\partial A^k_p} g^{k\bar{q}}$. It is well-known that $\frac{\partial \sigma_2}{\partial A^1_1} = A^2_2$, $\frac{\partial \sigma_2}{\partial A^2_2} = A^1_1$, and $\frac{\partial \sigma_2}{\partial A^1_2} = 0$ if A is diagonal. Then in our case,

$$\sigma_2^{1\bar{1}} = R_{22}, \quad \sigma_2^{2\bar{2}} = R_{11}, \quad \sigma_2^{1\bar{2}} = \sigma_2^{2\bar{1}} = 0. \quad (4.21)$$

We obtain

$$\begin{aligned} F^{1\bar{1}} &= 1 + \alpha' \|\Omega\|^3 \tilde{\rho}^{1\bar{1}} - \frac{\alpha'}{2} R_{22}, \quad F^{2\bar{2}} = 1 + \alpha' \|\Omega\|^3 \tilde{\rho}^{2\bar{2}} - \frac{\alpha'}{2} R_{11}, \\ F^{1\bar{2}} &= \alpha' \|\Omega\|^3 \tilde{\rho}^{1\bar{2}}, \quad F^{2\bar{1}} = \alpha' \|\Omega\|^3 \tilde{\rho}^{2\bar{1}}. \end{aligned} \quad (4.22)$$

4.2 Norm of the torsion

We will compute

$$\partial_t |T|^2 = \partial_t \{g^{i\bar{j}} T_i \bar{T}_{\bar{j}}\}. \quad (4.23)$$

We have

$$\partial_t g^{i\bar{j}} = -g^{i\bar{\lambda}} g^{\gamma\bar{j}} \partial_t g_{\bar{\lambda}\gamma} = \frac{1}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) g^{i\bar{j}}. \quad (4.24)$$

Hence

$$\begin{aligned} \partial_t |T|^2 &= 2\text{Re}\langle \partial_t T, T \rangle \\ &\quad + \frac{|T|^2}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) \end{aligned} \quad (4.25)$$

Next, using the notation $|W|_{Fg}^2 = F^{p\bar{q}} g^{i\bar{j}} W_{pi} \bar{W}_{\bar{q}\bar{j}}$,

$$\begin{aligned} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |T|^2 &= F^{p\bar{q}} g^{i\bar{j}} \nabla_p \nabla_{\bar{q}} T_i \bar{T}_{\bar{j}} + F^{p\bar{q}} g^{i\bar{j}} T_i \nabla_p \nabla_{\bar{q}} \bar{T}_{\bar{j}} + |\nabla T|_{Fg}^2 + |\bar{\nabla} T|_{Fg}^2 \\ &= F^{p\bar{q}} g^{i\bar{j}} \nabla_p \nabla_{\bar{q}} T_i \bar{T}_{\bar{j}} + g^{i\bar{j}} T_i \overline{F^{p\bar{q}} \nabla_{\bar{q}} \nabla_p T_j} + F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}pj}{}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} \\ &\quad + |\nabla T|_{Fg}^2 + |\bar{\nabla} T|_{Fg}^2. \end{aligned} \quad (4.26)$$

We introduce the notation $\Delta_F = F^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$. We have shown

$$\Delta_F |T|^2 = 2\text{Re}\langle \Delta_F T, T \rangle + |\nabla T|_{Fg}^2 + |\bar{\nabla} T|_{Fg}^2 + F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}pj}{}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}}. \quad (4.27)$$

Combining (4.20), (4.25), and (4.27), we obtain

$$\begin{aligned} \partial_t |T|^2 &= \frac{1}{2\|\Omega\|} \left\{ \Delta_F |T|^2 - |\nabla T|_{Fg}^2 - |\bar{\nabla} T|_{Fg}^2 - 2\text{Re}\{g^{i\bar{j}} \nabla_i |T|^2 \bar{T}_{\bar{j}}\} \right. \\ &\quad - \frac{1}{2} R |T|^2 + \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) |T|^2 + \text{Re}\{F^{p\bar{q}} g^{i\bar{j}} T_{pi}{}^\lambda R_{\bar{q}\lambda} \bar{T}_{\bar{j}}\} \\ &\quad - F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}pj}{}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} + |T|^4 + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} |T|^2 \\ &\quad \left. - \|\Omega\|^2 |T|^2 \nu + 2\text{Re}\langle E, T \rangle \right\}. \end{aligned} \quad (4.28)$$

4.3 Estimating the torsion

Theorem 4 *There exists $M_0 \gg 1$ such that all $M \geq M_0$ have the following property. Start the flow with a constant function $u_0 = \log M$. If*

$$|\alpha' \text{Ric}_\omega| \leq 10^{-6} \quad (4.29)$$

along the flow, then there exists $C_3 > 0$ depending only on $(X, \hat{\omega})$, ρ , $\tilde{\mu}$ and α' , such that

$$|T|^2 \leq \frac{C_3}{M^{4/3}} \ll 1. \quad (4.30)$$

Denote $\Lambda = 1 + \frac{1}{8}$. We will study the test function

$$G = \log |T|^2 - \Lambda \log \|\Omega\|. \quad (4.31)$$

Taking the time derivative gives us

$$\partial_t G = \frac{\partial_t |T|^2}{|T|^2} - \Lambda \partial_t \log \|\Omega\|. \quad (4.32)$$

Computing using (2.36) and (4.19),

$$\begin{aligned} \Delta_F \log \|\Omega\| &= F^{p\bar{q}} \nabla_p \bar{T}_{\bar{q}} = \frac{1}{2} F^{p\bar{q}} R_{\bar{q}p} \\ &= \frac{1}{2} R - \frac{\alpha'}{4} \sigma_2^{p\bar{q}} R_{\bar{q}p} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} \\ &= \frac{1}{2} R - \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}. \end{aligned} \quad (4.33)$$

Therefore by (4.11)

$$\partial_t \log \|\Omega\| = \frac{1}{2\|\Omega\|} \left\{ \Delta_F \log \|\Omega\| - |T|^2 + \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - \|\Omega\|^2 \nu \right\}. \quad (4.34)$$

Substituting (4.28) and (4.34) into (4.32), we have

$$\begin{aligned} \partial_t G &= \frac{1}{2\|\Omega\|} \left\{ \Delta_F G + \frac{|\nabla |T|^2|_F^2}{|T|^4} - \frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{|\bar{\nabla} T|_{Fg}^2}{|T|^2} - \frac{2}{|T|^2} \text{Re}\{g^{i\bar{j}} \nabla_i |T|^2 \bar{T}_{\bar{j}}\} \right. \\ &\quad - \frac{1}{2} R + \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) + \frac{1}{|T|^2} \text{Re}\{F^{p\bar{q}} g^{i\bar{j}} T_{\bar{p}i} R_{\bar{q}\lambda} \bar{T}_{\bar{j}}\} \\ &\quad - \frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}p\bar{j}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} + |T|^2 + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \|\Omega\|^2 \nu \\ &\quad \left. + \frac{2}{|T|^2} \text{Re}\langle E, T \rangle + \Lambda |T|^2 - \frac{\alpha'}{4} \Lambda \sigma_2(i\text{Ric}_\omega) + \Lambda \|\Omega\|^2 \nu \right\}. \end{aligned} \quad (4.35)$$

Let (p, t_0) be the point in $X \times [0, T]$ where G attains its maximum. Since we start the flow at $t = 0$ with a constant function $u_0 = \log M$, the torsion is zero at the initial time. It follows that $t_0 > 0$. The following computation will be done at this point (p, t_0) , and we note that $|T|^2 > 0$ at (p, t_0) . The critical equation $\nabla G = 0$ gives

$$0 = \frac{\nabla_i |T|^2}{|T|^2} - \Lambda T_i. \quad (4.36)$$

Using (2.44), this can be rewritten in the following way

$$\frac{\langle \nabla_i T, T \rangle}{|T|^2} = \Lambda T_i - \frac{\langle T, \nabla_{\bar{i}} T \rangle}{|T|^2} = \Lambda T_i - \frac{1}{2|T|^2} g^{j\bar{k}} T_j R_{\bar{k}i}. \quad (4.37)$$

Therefore, by Cauchy-Schwarz and the critical equation,

$$\begin{aligned} -\frac{|\nabla T|_{Fg}^2}{|T|^2} &\leq -\left|\frac{\langle \nabla T, T \rangle}{|T|^2}\right|_F^2 = -\left|\Lambda T_i - \frac{1}{2|T|^2} g^{j\bar{k}} T_j R_{\bar{k}i}\right|_F^2 \\ &= -\Lambda^2 |T|_F^2 - \frac{1}{4|T|^4} \left|g^{j\bar{k}} T_j R_{\bar{k}i}\right|_F^2 + \frac{\Lambda}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{j\bar{k}} T_j R_{\bar{k}p} \bar{T}_{\bar{q}}\}. \end{aligned} \quad (4.38)$$

We may also expand the following term using the definition of $F^{p\bar{q}}$,

$$4|\bar{\nabla} T|_{Fg}^2 = F^{p\bar{q}} g^{i\bar{j}} R_{\bar{q}i} R_{\bar{j}p} = |\operatorname{Ric}_\omega|^2 - \frac{\alpha'}{2} g^{i\bar{j}} \sigma_2^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + \alpha' \|\Omega\|^3 g^{i\bar{j}} \tilde{\rho}^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p}. \quad (4.39)$$

Set $\varepsilon = 1/100$. Using (4.38) and (4.39), and the critical equation (4.36) once more on the first and last term, we obtain

$$\begin{aligned} &\frac{|\nabla |T|^2|_F^2}{|T|^4} - (1-\varepsilon) \frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{|\bar{\nabla} T|_{Fg}^2}{|T|^2} - \frac{2}{|T|^2} \operatorname{Re}\{g^{i\bar{j}} \nabla_i |T|^2 \bar{T}_{\bar{j}}\} \\ &\leq \Lambda^2 |T|_F^2 - (1-\varepsilon) \Lambda^2 |T|_F^2 - (1-\varepsilon) \frac{1}{4|T|^4} \left|g^{j\bar{k}} T_j R_{\bar{k}i}\right|_F^2 \\ &\quad + (1-\varepsilon) \frac{\Lambda}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{j\bar{k}} T_j R_{\bar{k}p} \bar{T}_{\bar{q}}\} - \frac{1}{4} \frac{|\operatorname{Ric}_\omega|^2}{|T|^2} + \frac{\alpha'}{8|T|^2} g^{i\bar{j}} \sigma_2^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} \\ &\quad - \frac{\alpha'}{4|T|^2} \|\Omega\|^3 g^{i\bar{j}} \tilde{\rho}^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} - 2\Lambda |T|^2. \end{aligned} \quad (4.40)$$

Substituting this inequality into (4.35), our main inequality becomes

$$\begin{aligned} \partial_t G &\leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \varepsilon \frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{1}{4} \frac{|\operatorname{Ric}_\omega|^2}{|T|^2} - (\Lambda-1)|T|^2 + \varepsilon \Lambda^2 |T|_F^2 - \frac{1}{2} R \right. \\ &\quad - \frac{\alpha'}{4} (\Lambda-1) \sigma_2(i\operatorname{Ric}_\omega) + \frac{\alpha'}{8|T|^2} g^{i\bar{j}} \sigma_2^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + (1-\varepsilon) \frac{\Lambda}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{j\bar{k}} T_j R_{\bar{k}p} \bar{T}_{\bar{q}}\} \\ &\quad - \frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}p\bar{j}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} + \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{i\bar{j}} T_{pi} R_{\bar{q}\lambda} \bar{T}_{\bar{j}}\} \\ &\quad - \frac{(1-\varepsilon)}{4|T|^4} \left|g^{j\bar{k}} T_j R_{\bar{k}i}\right|_F^2 - \frac{\alpha'}{4|T|^2} \|\Omega\|^3 g^{i\bar{j}} \tilde{\rho}^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} \\ &\quad \left. + (\Lambda-1) \|\Omega\|^2 \nu + \frac{2}{|T|^2} \operatorname{Re}\langle E, T \rangle \right\}, \end{aligned} \quad (4.41)$$

which holds at (p, t_0) . Next, we use (4.2) to write the evolving curvature as

$$R_{\bar{q}p\bar{j}}^{\bar{\lambda}} = \hat{R}_{\bar{q}p\bar{j}}^{\bar{\lambda}} + \frac{1}{2} R_{\bar{q}p} \delta_j^{\bar{\lambda}}. \quad (4.42)$$

This identity allows us to write

$$-\frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}p\bar{j}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} = -\frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i \hat{R}_{\bar{q}p\bar{j}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} - \frac{1}{2} F^{p\bar{q}} R_{\bar{q}p}. \quad (4.43)$$

Next, by (4.3), the torsion can be written as

$$T^\lambda_{pi} = T_i \delta^\lambda_p - T_p \delta^\lambda_i, \quad (4.44)$$

so we may rewrite

$$\frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{i\bar{j}} T^\lambda_{pi} R_{\bar{q}\lambda} \bar{T}_{\bar{j}}\} = F^{p\bar{q}} R_{\bar{q}p} - \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{i\bar{j}} R_{\bar{q}i} \bar{T}_{\bar{j}} T_p\}. \quad (4.45)$$

Together, we have

$$\begin{aligned} & -\frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}p\bar{j}} \bar{T}_{\bar{\lambda}} + \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{i\bar{j}} T^\lambda_{pi} R_{\bar{q}\lambda} \bar{T}_{\bar{j}}\} \\ = & -\frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i \hat{R}_{\bar{q}p\bar{j}} \bar{T}_{\bar{\lambda}} + \frac{1}{2} F^{p\bar{q}} R_{\bar{q}p} - \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{i\bar{j}} R_{\bar{q}i} \bar{T}_{\bar{j}} T_p\} \\ = & -\frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i \hat{R}_{\bar{q}p\bar{j}} \bar{T}_{\bar{\lambda}} + \frac{1}{2} R - \frac{1}{2} \alpha' \sigma_2(i \operatorname{Ric}_\omega) \\ & + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{i\bar{j}} R_{\bar{q}i} \bar{T}_{\bar{j}} T_p\}. \end{aligned} \quad (4.46)$$

We also compute

$$|T|_F^2 = |T|^2 + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}} T_p \bar{T}_{\bar{q}}. \quad (4.47)$$

Substituting (4.46) and (4.47) in the main inequality (4.41), we see that the terms of order R have cancelled.

$$\begin{aligned} \partial_t G \leq & \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \varepsilon \frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{1}{4} \frac{|\operatorname{Ric}_\omega|^2}{|T|^2} - (\Lambda - 1 - \varepsilon \Lambda^2) |T|^2 \right. \\ & - \varepsilon \Lambda^2 \frac{\alpha'}{2} \sigma_2^{p\bar{q}} T_p \bar{T}_{\bar{q}} - (1 + \Lambda) \frac{\alpha'}{4} \sigma_2(i \operatorname{Ric}_\omega) + \frac{\alpha'}{8|T|^2} g^{i\bar{j}} \sigma_2^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} \\ & + (\Lambda - \varepsilon \Lambda - 1) \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{j\bar{k}} T_j R_{\bar{k}p} \bar{T}_{\bar{q}}\} - \frac{(1 - \varepsilon)}{4|T|^4} \left| g^{j\bar{k}} T_j R_{\bar{k}i} \right|_F^2 \\ & - \frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i \hat{R}_{\bar{q}p\bar{j}} \bar{T}_{\bar{\lambda}} - \frac{\alpha'}{4|T|^2} \|\Omega\|^3 g^{i\bar{j}} \tilde{\rho}^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + \varepsilon \Lambda^2 \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} \\ & \left. + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + (\Lambda - 1) \|\Omega\|^2 \nu + \frac{2}{|T|^2} \operatorname{Re}\langle E, T \rangle \right\}. \end{aligned} \quad (4.48)$$

We now substitute $\Lambda = 1 + \frac{1}{8}$ and $\varepsilon = \frac{1}{100}$. Then

$$\begin{aligned} \partial_t G \leq & \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \frac{1}{100} \frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{1}{4} \frac{|\operatorname{Ric}_\omega|^2}{|T|^2} - \frac{1}{9} |T|^2 - \left(\frac{9}{8}\right)^2 \frac{1}{100} \frac{\alpha'}{2} \sigma_2^{p\bar{q}} T_p \bar{T}_{\bar{q}} \right. \\ & - \frac{17}{16} \frac{\alpha'}{2} \sigma_2(i \operatorname{Ric}_\omega) + \frac{\alpha'}{8|T|^2} g^{i\bar{j}} \sigma_2^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + \left(\frac{1}{8} - \frac{9}{800}\right) \frac{1}{|T|^2} \operatorname{Re}\{F^{p\bar{q}} g^{j\bar{k}} T_j R_{\bar{k}p} \bar{T}_{\bar{q}}\} \\ & \left. + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + \left(\frac{1}{8} - \frac{9}{800}\right) \frac{1}{|T|^2} \operatorname{Re}\langle E, T \rangle \right\}. \end{aligned}$$

$$\begin{aligned}
& -\frac{99}{400} \frac{1}{|T|^4} \left| g^{j\bar{k}} T_j R_{\bar{k}i} \right|_F^2 - \frac{1}{|T|^2} F^{p\bar{q}} g^{i\bar{j}} T_i \hat{R}_{\bar{q}p\bar{j}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} - \frac{\alpha'}{4|T|^2} \|\Omega\|^3 g^{i\bar{j}} \tilde{\rho}^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} \\
& + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + \frac{1}{100} \left(\frac{9}{8} \right)^2 \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} + \frac{1}{8} \|\Omega\|^2 \nu + \frac{2}{|T|^2} \text{Re}\langle E, T \rangle \} \quad (4.49)
\end{aligned}$$

We are assuming in the hypothesis of Theorem 4 that $|\alpha' \text{Ric}_\omega| < 10^{-6}$. By Theorem 3, we know that $\|\Omega\| \leq \frac{C_2}{M} \ll 1$, so for M large enough we can assume

$$(1 - 10^{-6}) g^{i\bar{j}} \leq F^{i\bar{j}} \leq (1 + 10^{-6}) g^{i\bar{j}}. \quad (4.50)$$

One way to see this inequality is by writing $F^{i\bar{j}}$ in coordinates (4.22). Using (4.50), we can estimate

$$\begin{aligned}
& -\frac{17}{16} \frac{\alpha'}{2} \sigma_2(i \text{Ric}_\omega) - \frac{\alpha'}{8|T|^2} g^{i\bar{j}} \sigma_2^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + \left(\frac{1}{8} - \frac{9}{800} \right) \frac{1}{|T|^2} \text{Re}\{ F^{p\bar{q}} g^{j\bar{k}} T_j R_{\bar{k}p} \bar{T}_{\bar{q}} \} \\
& \leq \frac{17}{16} \frac{1}{2} |\alpha' \text{Ric}_\omega| |\text{Ric}_\omega| + \frac{1}{8} |\alpha' \text{Ric}_\omega| \frac{|\text{Ric}_\omega|^2}{|T|^2} + \frac{1}{7} |\text{Ric}_\omega| \\
& \leq \frac{1}{(2)(3)} |\text{Ric}_\omega| + \frac{1}{100} \frac{|\text{Ric}_\omega|^2}{|T|^2} \\
& \leq \frac{1}{(2)(3)^2} |T|^2 + \left(\frac{1}{100} + \frac{1}{(2)(2)^2} \right) \frac{|\text{Ric}_\omega|^2}{|T|^2}. \quad (4.51)
\end{aligned}$$

We also notice

$$-\left(\frac{9}{8} \right)^2 \frac{1}{100} \frac{\alpha'}{2} \sigma_2^{p\bar{q}} T_p \bar{T}_{\bar{q}} \leq |\alpha' \text{Ric}| |T|^2 \leq \frac{1}{10^6} |T|^2, \quad (4.52)$$

and

$$|\hat{R}m|_g^2 = e^{-4u} |\hat{R}m|_{\hat{g}}^2 = \|\Omega\|^4 |\hat{R}m|_{\hat{g}}^2. \quad (4.53)$$

Substituting these estimates into (4.49) gives

$$\begin{aligned}
\partial_t G & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \frac{1}{200} \frac{|\nabla T|^2}{|T|^2} - \frac{1}{100} \frac{|\text{Ric}_\omega|^2}{|T|^2} - \frac{1}{100} |T|^2 \right. \\
& + 2\|\Omega\|^2 |\hat{R}m|_{\hat{g}} + \frac{\alpha'}{4|T|^2} \|\Omega\|^3 g^{i\bar{j}} \tilde{\rho}^{p\bar{q}} R_{\bar{q}i} R_{\bar{j}p} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} \\
& \left. + \frac{1}{100} \left(\frac{9}{8} \right)^2 \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} + \frac{1}{8} \|\Omega\|^2 \nu + \frac{2}{|T|^2} \text{Re}\langle E, T \rangle \right\}. \quad (4.54)
\end{aligned}$$

By the definition of E (4.14) and ν (4.10), the terms on the last two lines can only slightly perturb the coefficients of the first line since $\|\Omega\| = e^{-u} \leq \frac{C_2}{M} \ll 1$ for $M \gg 1$ large enough. We recall that $\tilde{\rho}^{p\bar{q}}$ and b_ρ^i are bounded in C^∞ in terms of the background metric \hat{g} , so for example,

$$\|\Omega\| \tilde{\rho}^{p\bar{q}} \leq C e^{-u} \hat{g}^{p\bar{q}} = C g^{p\bar{q}}, \quad \|\Omega\|^{1/2} |b_\rho^i T_i| \leq C |T|. \quad (4.55)$$

This allows us to bound certain terms such as

$$\alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} \leq C \|\Omega\|^2 |\text{Ric}_\omega| \leq \frac{C}{2} \|\Omega\|^2 \frac{|\text{Ric}_\omega|^2}{|T|^2} + \frac{C}{2} \|\Omega\|^2 |T|^2, \quad (4.56)$$

and

$$\alpha' \|\Omega\|^3 \text{Re}\{b_\rho^i T_i\} \leq C \|\Omega\|^2 |T| \leq C \|\Omega\|^2 \frac{|T|^2}{2} + \frac{C}{2} \|\Omega\|^2. \quad (4.57)$$

Covariant derivatives with respect to the evolving metric act like $\nabla_i = \partial_i - T_i$, so we can bound terms such as

$$\frac{2}{|T|^2} \frac{\alpha'}{2} \|\Omega\|^3 g^{j\bar{k}} (\nabla_j \tilde{\rho}^{p\bar{q}}) R_{\bar{q}p} \bar{T}_k \leq C \|\Omega\|^2 \frac{|\text{Ric}_\omega|}{|T|} + C \|\Omega\|^2 \frac{|\text{Ric}_\omega|}{|T|} |T|. \quad (4.58)$$

The inequality $2ab \leq a^2 + b^2$ can be used to absorb terms into the first line. Using these estimates, it is possible to show that at the maximum point (p, t_0) of G , for $\|\Omega\| \leq \frac{C_2}{M} \ll 1$, there holds

$$0 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \frac{1}{200} |T|^2 + C \|\Omega\|^2 \left(1 + \frac{1}{|T|}\right) \right\}. \quad (4.59)$$

By (4.50), $\Delta_F G \leq 0$ at the maximum (p, t_0) of G , hence

$$|T|^3 \leq C \|\Omega\|^2 (1 + |T|) \leq \frac{1}{2} |T|^3 + C \|\Omega\|^2 + C \|\Omega\|^3. \quad (4.60)$$

Hence

$$|T|^3 \leq \frac{C}{M^2}, \quad (4.61)$$

Therefore

$$G \leq G(p, t_0) \leq \log \frac{C}{M^{4/3}} + \Lambda u(p). \quad (4.62)$$

By Theorem 3,

$$\begin{aligned} |T|^2 &\leq \frac{C}{M^{4/3}} \exp \{ \Lambda(u(p) - u) \} \\ &\leq \frac{C}{M^{4/3}} \left(\sup_{X \times [0, T)} e^u \right)^\Lambda \left(\sup_{X \times [0, T)} e^{-u} \right)^\Lambda \\ &\leq \frac{C}{M^{4/3}} (C_2 C_1)^\Lambda \ll 1. \end{aligned} \quad (4.63)$$

This proves Theorem 4.

5 Evolution of the curvature

5.1 Ricci curvature

In this subsection, we flow the Ricci curvature of the evolving Hermitian metric $e^u \hat{g}$. We will use the well-known general formula for the evolution of the curvature tensor

$$\partial_t R_{\bar{k}j}{}^\alpha{}_\beta = -\nabla_{\bar{k}} \nabla_j (g^{\alpha\bar{\gamma}} \partial_t g_{\bar{\gamma}\beta}). \quad (5.1)$$

Recall that we defined $R_{\bar{k}j} = R_{\bar{k}j}{}^\alpha{}_\alpha$, hence substituting (1.1) yields

$$\partial_t R_{\bar{k}j} = -\nabla_{\bar{k}} \nabla_j \left\{ \frac{1}{2\|\Omega\|} \left(-R - \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) + 2|T|^2 + 2\|\Omega\|^2 \nu \right) \right\}. \quad (5.2)$$

Expanding out terms gives

$$\begin{aligned} \partial_t R_{\bar{k}j} &= \frac{1}{2\|\Omega\|} \left\{ \nabla_{\bar{k}} \nabla_j R + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_{\bar{k}} \nabla_j R_{\bar{q}p} - \nabla_{\bar{k}} \nabla_j \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) - 2\nabla_{\bar{k}} \nabla_j |T|^2 \right. \\ &\quad + \alpha' \nabla_{\bar{k}} (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_j R_{\bar{q}p} + \alpha' \nabla_j (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_{\bar{k}} R_{\bar{q}p} \\ &\quad + \alpha' \nabla_{\bar{k}} \nabla_j (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) R_{\bar{q}p} - \nabla_{\bar{k}} \nabla_j 2\|\Omega\|^2 \nu \left. \right\} \\ &\quad - \frac{\nabla_j \|\Omega\|}{2\|\Omega\|^2} \nabla_{\bar{k}} \left\{ R + \alpha' (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) - \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2 \nu \right\} \\ &\quad - \frac{\nabla_{\bar{k}} \|\Omega\|}{2\|\Omega\|^2} \nabla_j \left\{ R + \alpha' (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) - \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2 \nu \right\} \\ &\quad + \left\{ \frac{-\nabla_{\bar{k}} \nabla_j \|\Omega\|}{2\|\Omega\|^2} + \frac{2\nabla_{\bar{k}} \|\Omega\| \nabla_j \|\Omega\|}{2\|\Omega\|^3} \right\} \left\{ R + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} \right. \\ &\quad \left. - \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2 \nu \right\}. \end{aligned} \quad (5.3)$$

Using $\nabla_j \|\Omega\| = \|\Omega\| T_j$,

$$\begin{aligned} \partial_t R_{\bar{k}j} &= \frac{1}{2\|\Omega\|} \left\{ \nabla_{\bar{k}} \nabla_j R + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_{\bar{k}} \nabla_j R_{\bar{q}p} - \nabla_{\bar{k}} \nabla_j \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) \right. \\ &\quad - 2\nabla_{\bar{k}} \nabla_j |T|^2 + \alpha' \nabla_{\bar{k}} (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_j R_{\bar{q}p} + \alpha' \nabla_j (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_{\bar{k}} R_{\bar{q}p} \\ &\quad + \alpha' \nabla_{\bar{k}} \nabla_j (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) R_{\bar{q}p} - 2\nabla_{\bar{k}} \nabla_j \|\Omega\|^2 \nu \left. \right\} - T_j \nabla_{\bar{k}} R \\ &\quad - \alpha' T_j \nabla_{\bar{k}} (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) + 2T_j \nabla_{\bar{k}} |T|^2 + \frac{\alpha'}{2} T_j \nabla_{\bar{k}} (\sigma_2(i\text{Ric}_\omega)) + 2T_j \nabla_{\bar{k}} \|\Omega\|^2 \nu \\ &\quad - T_{\bar{k}} \nabla_j R - \alpha' T_{\bar{k}} \nabla_j (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) + 2T_{\bar{k}} \nabla_j |T|^2 + \frac{\alpha'}{2} T_{\bar{k}} \nabla_j (\sigma_2(i\text{Ric}_\omega)) \\ &\quad + 2T_{\bar{k}} \nabla_j \|\Omega\|^2 \nu + R T_j T_{\bar{k}} + \alpha' T_j T_{\bar{k}} (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) - 2|T|^2 T_j T_{\bar{k}} \\ &\quad - \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) T_j T_{\bar{k}} - 2T_j T_{\bar{k}} \|\Omega\|^2 \nu - R \nabla_{\bar{k}} T_j - \alpha' \nabla_{\bar{k}} T_j (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) \\ &\quad + 2|T|^2 \nabla_{\bar{k}} T_j + \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) \nabla_{\bar{k}} T_j + 2\nabla_{\bar{k}} T_j \|\Omega\|^2 \nu \left. \right\}. \end{aligned} \quad (5.4)$$

We now study the highest order terms, namely

$$\nabla_{\bar{k}} \nabla_j R_{\bar{q}p} = -2 \nabla_{\bar{k}} \nabla_j \nabla_p \nabla_{\bar{q}} u. \quad (5.5)$$

We will use the following commutation formula for covariant derivatives in Hermitian geometry

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j \nabla_p \nabla_{\bar{q}} u &= \nabla_p \nabla_{\bar{q}} \nabla_j \nabla_{\bar{k}} u + T^\lambda_{pj} \nabla_{\bar{q}} \nabla_\lambda \nabla_{\bar{k}} u + \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} \nabla_p \nabla_j \nabla_{\bar{\lambda}} u \\ &\quad + R_{\bar{k}j}^\lambda u_{\bar{q}\lambda} - R_{\bar{q}p\bar{k}}^{\bar{\lambda}} u_{\bar{\lambda}j} + \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} T^\gamma_{pj} u_{\bar{\lambda}\gamma}. \end{aligned} \quad (5.6)$$

Using $R_{\bar{q}p} = -2u_{\bar{q}p}$, we obtain

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j R_{\bar{q}p} &= \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} + T^\lambda_{pj} \nabla_{\bar{q}} R_{\bar{k}\lambda} + \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} \nabla_p R_{\bar{\lambda}j} \\ &\quad + R_{\bar{k}j}^\lambda R_{\bar{q}\lambda} - R_{\bar{q}p\bar{k}}^{\bar{\lambda}} R_{\bar{\lambda}j} + \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} T^\gamma_{pj} R_{\bar{\lambda}\gamma}. \end{aligned} \quad (5.7)$$

Hence

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j R &= g^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} + g^{p\bar{q}} T^\lambda_{pj} \nabla_{\bar{q}} R_{\bar{k}\lambda} + g^{p\bar{q}} \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} \nabla_p R_{\bar{\lambda}j} \\ &\quad + R_{\bar{k}j}^{p\bar{q}} R_{\bar{q}p} - R_{p\bar{k}}^{\bar{\lambda}} R_{\bar{\lambda}j} + g^{p\bar{q}} \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} T^\gamma_{pj} R_{\bar{\lambda}\gamma}. \end{aligned} \quad (5.8)$$

Next,

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j \sigma_2(i\text{Ric}_\omega) &= \sigma_2^{p\bar{q}} \nabla_{\bar{k}} \nabla_j R_{\bar{q}p} + \sigma_2^{p\bar{q}, r\bar{s}} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p} \\ &= \sigma_2^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} + \sigma_2^{p\bar{q}, r\bar{s}} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p} + \sigma_2^{p\bar{q}} T^\lambda_{pj} \nabla_{\bar{q}} R_{\bar{k}\lambda} \\ &\quad + \sigma_2^{p\bar{q}} \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} \nabla_p R_{\bar{\lambda}j} + \sigma_2^{p\bar{q}} R_{\bar{k}j}^\lambda R_{\bar{q}\lambda} - \sigma_2^{p\bar{q}} R_{\bar{q}p\bar{k}}^{\bar{\lambda}} R_{\bar{\lambda}j} \\ &\quad + \sigma_2^{p\bar{q}} \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} T^\gamma_{pj} R_{\bar{\lambda}\gamma}. \end{aligned} \quad (5.9)$$

Let us recall the definition of $\sigma_2^{p\bar{q}, r\bar{s}}$. For a Hermitian endomorphism H , we define $\sigma_2(H) = \sigma_2(\lambda)$ to be the second elementary symmetric polynomial of its eigenvalues. Our operator $\sigma_2(i\text{Ric}_\omega)$ defined in (2.42) is with respect to the evolving metric ω , so denoting $A^i_j = g^{i\bar{k}} R_{\bar{k}j}$, we have $\sigma_2(i\text{Ric}_\omega) = \sigma_2(A)$ and

$$\sigma_2^{p\bar{q}, r\bar{s}}(A) = g^{i\bar{q}} g^{j\bar{s}} \frac{\partial^2 \sigma_2}{\partial A^j_r \partial A^i_p}(A). \quad (5.10)$$

It is well-known that $\frac{\partial^2 \sigma_2}{\partial A^j_r \partial A^i_p} = \pm 1$. By (5.8) and (5.9), and proceeding similarly for the ρ term, we obtain

$$\begin{aligned} &\nabla_{\bar{k}} \nabla_j R + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_{\bar{k}} \nabla_j R_{\bar{q}p} - \frac{\alpha'}{2} \nabla_{\bar{k}} \nabla_j \sigma_2(i\text{Ric}_\omega) \\ &= F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p} + F^{p\bar{q}} T^\lambda_{pj} \nabla_{\bar{q}} R_{\bar{k}\lambda} + F^{p\bar{q}} \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} \nabla_p R_{\bar{\lambda}j} \\ &\quad + F^{p\bar{q}} R_{\bar{k}j}^\lambda R_{\bar{q}\lambda} - F^{p\bar{q}} R_{\bar{q}p\bar{k}}^{\bar{\lambda}} R_{\bar{\lambda}j} + F^{p\bar{q}} \bar{T}^{\bar{\lambda}}_{\bar{q}\bar{k}} T^\gamma_{pj} R_{\bar{\lambda}\gamma}, \end{aligned} \quad (5.11)$$

where the definition of $F^{p\bar{q}}$ was given in (4.19).

Using (2.44), we may convert derivatives of torsion $\bar{\nabla}T$ into curvature terms, but terms ∇T are of different type and must be treated separately. For example

$$\begin{aligned}
-2\nabla_{\bar{k}}\nabla_j|T|^2 &= -2g^{p\bar{q}}\nabla_{\bar{k}}\nabla_jT_p\bar{T}_{\bar{q}} - 2g^{p\bar{q}}\nabla_jT_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} - \frac{1}{2}g^{p\bar{q}}R_{\bar{k}p}R_{\bar{q}j} - g^{p\bar{q}}T_p\nabla_{\bar{k}}R_{\bar{q}j} \\
&= -g^{p\bar{q}}\nabla_jR_{\bar{k}p}\bar{T}_{\bar{q}} - 2g^{p\bar{q}}R_{\bar{k}j}{}^\lambda{}_pT_\lambda\bar{T}_{\bar{q}} - 2g^{p\bar{q}}\nabla_jT_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} \\
&\quad - \frac{1}{2}g^{p\bar{q}}R_{\bar{k}p}R_{\bar{q}j} - g^{p\bar{q}}T_p\nabla_{\bar{k}}R_{\bar{q}j}.
\end{aligned} \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.4),

$$\begin{aligned}
\partial_t R_{\bar{k}j} &= \frac{1}{2\|\Omega\|} \left\{ F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}R_{\bar{k}j} - \frac{\alpha'}{2}\sigma_2^{p\bar{q},r\bar{s}}\nabla_{\bar{k}}R_{\bar{s}r}\nabla_jR_{\bar{q}p} \right. \\
&\quad \left. + 2\alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}\nabla_jT_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} - 2g^{p\bar{q}}\nabla_jT_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} + Y_{\bar{k}j} \right\}.
\end{aligned} \tag{5.13}$$

where $Y_{\bar{k}j}$ contains various combinations of torsion and curvature terms, but is linear in first derivatives of curvature and torsion and does not contain higher order derivatives of curvature and torsion. Explicitly,

$$\begin{aligned}
Y_{\bar{k}j} &= F^{p\bar{q}}T^\lambda{}_{pj}\nabla_{\bar{q}}R_{\bar{k}\lambda} + F^{p\bar{q}}\bar{T}^{\bar{\lambda}}{}_{\bar{q}\bar{k}}\nabla_pR_{\bar{\lambda}j} + F^{p\bar{q}}R_{\bar{k}j}{}^\lambda{}_pR_{\bar{q}\lambda} - F^{p\bar{q}}R_{\bar{q}p\bar{k}}{}^{\bar{\lambda}}R_{\bar{\lambda}j} \\
&\quad + F^{p\bar{q}}\bar{T}^{\bar{\lambda}}{}_{\bar{q}\bar{k}}T^\gamma{}_{pj}R_{\bar{\lambda}\gamma} - g^{p\bar{q}}\nabla_jR_{\bar{k}p}\bar{T}_{\bar{q}} - 2g^{p\bar{q}}R_{\bar{k}j}{}^\lambda{}_pT_\lambda\bar{T}_{\bar{q}} - \frac{1}{2}g^{p\bar{q}}R_{\bar{k}p}R_{\bar{q}j} \\
&\quad - g^{p\bar{q}}T_p\nabla_{\bar{k}}R_{\bar{q}j} + \alpha'\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_jR_{\bar{q}p} + \alpha'\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_{\bar{k}}R_{\bar{q}p} \\
&\quad + \alpha'(\nabla_{\bar{k}}\nabla_j\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} - T_jF^{p\bar{q}}\nabla_{\bar{k}}R_{\bar{q}p} - \alpha'T_j\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} + g^{p\bar{q}}T_jR_{\bar{k}p}\bar{T}_{\bar{q}} \\
&\quad + 2g^{p\bar{q}}T_jT_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} - 2\left\{ -\alpha'\nabla_{\bar{k}}\nabla_j(\|\Omega\|^3\psi_\rho) - \frac{\alpha'}{2}\text{Re}\{\|\Omega\|^3b_\rho^i\nabla_jR_{\bar{k}i}\} \right. \\
&\quad - \alpha'\text{Re}\{\|\Omega\|^3b_\rho^iR_{\bar{k}j}{}^\lambda{}_iT_\lambda\} - \alpha'\text{Re}\{\nabla_{\bar{k}}(\|\Omega\|^3b_\rho^i)\nabla_jT_i\} \\
&\quad \left. - \alpha'\text{Re}\{\nabla_j(\|\Omega\|^3b_\rho^i)\nabla_{\bar{k}}T_i\} - \alpha'\text{Re}\{\nabla_{\bar{k}}\nabla_j(\|\Omega\|^3b_\rho^i)T_i\} + \nabla_{\bar{k}}\nabla_j(\|\Omega\|^2\tilde{\mu}) \right\} \\
&\quad + \left\{ 2\alpha'(\nabla_{\bar{k}}\nabla_j\|\Omega\|^3\tilde{\rho}^{p\bar{q}})T_p\bar{T}_{\bar{q}} + 2\alpha'\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_j(T_p\bar{T}_{\bar{q}}) \right. \\
&\quad + 2\alpha'\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_{\bar{k}}(T_p\bar{T}_{\bar{q}}) + \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}\nabla_jR_{\bar{k}p}\bar{T}_{\bar{q}} + 2\alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{k}j}{}^\lambda{}_pT_\lambda\bar{T}_{\bar{q}} \\
&\quad \left. + \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\nabla_{\bar{k}}R_{\bar{q}j} + \frac{\alpha'}{2}\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{k}p}R_{\bar{q}j} \right\} \\
&\quad + 2T_j\nabla_{\bar{k}}\left\{ -\alpha'\|\Omega\|^3\psi_\rho - \alpha'\|\Omega\|^3\text{Re}\{b_\rho^iT_i\} - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\bar{T}_{\bar{q}} + \|\Omega\|^2\tilde{\mu} \right\} \\
&\quad - T_{\bar{k}}F^{p\bar{q}}\nabla_jR_{\bar{q}p} - \alpha'T_{\bar{k}}\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} + 2g^{p\bar{q}}T_{\bar{k}}\nabla_jT_p\bar{T}_{\bar{q}} + g^{p\bar{q}}T_{\bar{k}}T_pR_{\bar{q}j} \\
&\quad + 2T_{\bar{k}}\nabla_j\left\{ -\alpha'\|\Omega\|^3\psi_\rho - \alpha'\|\Omega\|^3\text{Re}\{b_\rho^iT_i\} - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\bar{T}_{\bar{q}} + \|\Omega\|^2\tilde{\mu} \right\} \\
&\quad + RT_jT_{\bar{k}} + \alpha'T_jT_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) - 2|T|^2T_jT_{\bar{k}} - \frac{\alpha'}{2}\sigma_2(i\text{Ric}_\omega)T_jT_{\bar{k}}
\end{aligned}$$

$$\begin{aligned}
& -2T_j T_{\bar{k}} \left\{ -\alpha' \|\Omega\|^3 \psi_\rho - \alpha' \|\Omega\|^3 \text{Re}\{b_\rho^i T_i\} - \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} + \|\Omega\|^2 \tilde{\mu} \right\} \\
& -\frac{1}{2} R R_{\bar{k}j} - \frac{\alpha'}{2} R_{\bar{k}j} (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p}) + |T|^2 R_{\bar{k}j} + \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) R_{\bar{k}j} \\
& + R_{\bar{k}j} \left\{ -\alpha' \|\Omega\|^3 \psi_\rho - \alpha' \|\Omega\|^3 \text{Re}\{b_\rho^i T_i\} - \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} T_p \bar{T}_{\bar{q}} + \|\Omega\|^2 \tilde{\mu} \right\}. \quad (5.14)
\end{aligned}$$

The terms in brackets indicate terms which come from substituting the definition of ν (4.10).

5.2 Evolving the norm of the curvature

We will compute

$$\partial_t |\text{Ric}_\omega|^2 = \partial_t \{g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} \overline{R_{\bar{k}j}}\}. \quad (5.15)$$

We have

$$\begin{aligned}
\partial_t g^{i\bar{j}} &= -g^{i\bar{\lambda}} g^{\gamma\bar{j}} \partial_t g_{\bar{\lambda}\gamma} \\
&= \frac{1}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) g^{i\bar{j}}. \quad (5.16)
\end{aligned}$$

Hence

$$\begin{aligned}
\partial_t |\text{Ric}_\omega|^2 &= 2\text{Re}\langle \partial_t \text{Ric}_\omega, \text{Ric}_\omega \rangle \\
&+ \frac{|\text{Ric}_\omega|^2}{2\|\Omega\|} \left(R + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{2} \sigma_2(i\text{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2 \nu \right). \quad (5.17)
\end{aligned}$$

Next,

$$\begin{aligned}
F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\text{Ric}_\omega|^2 &= g^{k\bar{\ell}} g^{i\bar{j}} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{\ell}i} R_{\bar{j}k} + g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{j}k} \\
&+ |\nabla \text{Ric}_\omega|_{Fgg}^2 + |\overline{\nabla} \text{Ric}_\omega|_{Fgg}^2 \\
&= g^{k\bar{\ell}} g^{i\bar{j}} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{\ell}i} \overline{R_{\bar{k}j}} + g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} \overline{F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j}} \\
&- g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} R_{\bar{q}p}{}^\lambda R_{\bar{j}\lambda} + g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}} R_{\bar{\lambda}k} \\
&+ |\nabla \text{Ric}_\omega|_{Fgg}^2 + |\overline{\nabla} \text{Ric}_\omega|_{Fgg}^2. \quad (5.18)
\end{aligned}$$

We have shown

$$\begin{aligned}
\Delta_F |\text{Ric}_\omega|^2 &= 2\text{Re}\langle \Delta_F \text{Ric}_\omega, \text{Ric}_\omega \rangle + |\nabla \text{Ric}_\omega|_{Fgg}^2 + |\overline{\nabla} \text{Ric}_\omega|_{Fgg}^2 \\
&- g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} R_{\bar{q}p}{}^\lambda R_{\bar{j}\lambda} + g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}} R_{\bar{\lambda}k}. \quad (5.19)
\end{aligned}$$

Substituting (5.13) into (5.17) gives

$$\partial_t |\text{Ric}_\omega|^2 = \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\text{Ric}_\omega|^2 - |\nabla \text{Ric}_\omega|_{Fgg}^2 - |\overline{\nabla} \text{Ric}_\omega|_{Fgg}^2 \right\} \quad (5.20)$$

$$\begin{aligned}
& -\alpha' \operatorname{Re}\{g^{j\bar{\ell}} g^{m\bar{k}} \sigma_2^{p\bar{q}, r\bar{s}} R_{\bar{\ell}m} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p}\} \\
& + 4\alpha' \operatorname{Re}\{g^{j\bar{\ell}} g^{m\bar{k}} R_{\bar{\ell}m} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j T_p \nabla_{\bar{k}} \bar{T}_{\bar{q}}\} \\
& - 4\operatorname{Re}\{g^{j\bar{\ell}} g^{m\bar{k}} R_{\bar{\ell}m} g^{p\bar{q}} \nabla_j T_p \nabla_{\bar{k}} \bar{T}_{\bar{q}}\} + 2\operatorname{Re}\{g^{j\bar{\ell}} g^{m\bar{k}} R_{\bar{\ell}m} Y_{\bar{k}j}\} \\
& + g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} R_{\bar{q}p}{}^\lambda{}_k R_{\bar{j}\lambda} - g^{k\bar{\ell}} g^{i\bar{j}} R_{\bar{\ell}i} F^{p\bar{q}} R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}} R_{\bar{\lambda}k} + |\operatorname{Ric}_\omega|^2 R \\
& + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} |\operatorname{Ric}_\omega|^2 - 2|T|^2 |\operatorname{Ric}_\omega|^2 - \frac{\alpha'}{2} \sigma_2(i\operatorname{Ric}_\omega) |\operatorname{Ric}_\omega|^2 \\
& - 2\|\Omega\|^2 |\operatorname{Ric}_\omega|^2 \nu \}.
\end{aligned}$$

5.3 Estimating Ricci curvature

Lemma 1 *Let $0 < \delta, \epsilon < \frac{1}{2}$ be such that $-\frac{1}{4}g^{p\bar{q}} < \alpha'\delta^2\|\Omega\|\tilde{\rho}^{p\bar{q}} < \frac{1}{4}g^{p\bar{q}}$, and*

$$\|\Omega\|^2 \leq \delta, \quad |T|^2 \leq \delta, \quad |\alpha' \operatorname{Ric}_\omega| \leq \epsilon, \quad (5.21)$$

at a point (p, t_0) . Let $\Lambda > 1$ be any constant. Then at (p, t_0) there holds

$$\begin{aligned}
& \partial_t(|\alpha' \operatorname{Ric}_\omega|^2 + \Lambda|T|^2) \\
& \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F(|\alpha' \operatorname{Ric}_\omega|^2 + \Lambda|T|^2) - \left(\frac{1}{2} - 2\epsilon\right) |\alpha' \nabla \operatorname{Ric}_\omega|^2 \right. \\
& \quad \left. - \left(\frac{\Lambda}{4} - (5 + C\delta^2)\epsilon|\alpha'|\right) |\nabla T|^2 - \frac{\Lambda}{8} |\operatorname{Ric}_\omega|^2 + C(1 + \Lambda)\epsilon\delta + C\epsilon^2 + C\Lambda\delta \right\}, \quad (5.22)
\end{aligned}$$

for some constant C only depending on μ, ρ, α' , and the background manifold $(X, \hat{\omega})$.

Proof: Since ϵ and δ are assumed to be small, we have

$$F^{p\bar{q}} = g^{p\bar{q}} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}}, \quad \frac{1}{2}g^{p\bar{q}} < F^{p\bar{q}} < \frac{3}{2}g^{p\bar{q}}. \quad (5.23)$$

In coordinates where g is the identity, it is well-known that $\sigma_2^{p\bar{q}, r\bar{s}} = \pm 1$. We note the following estimate

$$-\alpha' \operatorname{Re}\{g^{j\bar{\ell}} g^{m\bar{k}} \sigma_2^{p\bar{q}, r\bar{s}} R_{\bar{\ell}m} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p}\} \leq |\alpha' \operatorname{Ric}_\omega| |\nabla \operatorname{Ric}_\omega|^2. \quad (5.24)$$

We will estimate and group terms in (5.20) and (5.14). We will convert $F^{p\bar{q}}$ into the metric $g^{p\bar{q}}$, and handle $\tilde{\rho}^{p\bar{q}}$ and b^i as in (4.55). We will also use that the norm of the full torsion $T(\omega) = i\partial\omega$ is $2|T|^2$, $\nabla_i \|\Omega\| = \|\Omega\| T_i$, $\nabla_{\bar{k}} \nabla_i \|\Omega\| = \|\Omega\| T_i \bar{T}_{\bar{k}} + 2^{-1} \|\Omega\| R_{\bar{k}j}$, and $\|\Omega\| \leq 1$.

$$\begin{aligned}
\partial_t |\operatorname{Ric}_\omega|^2 & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\operatorname{Ric}_\omega|^2 - \frac{1}{2} |\nabla \operatorname{Ric}_\omega|^2 - \frac{1}{2} |\bar{\nabla} \operatorname{Ric}_\omega|^2 \right. \\
& \quad \left. + |\alpha' \operatorname{Ric}_\omega| |\nabla \operatorname{Ric}_\omega|^2 + (4 + C\|\Omega\|^2) |\operatorname{Ric}_\omega| |\nabla T|^2 \right\} \quad (5.25)
\end{aligned}$$

$$\begin{aligned}
& + \frac{C}{2\|\Omega\|} \left\{ |T| |\text{Ric}_\omega| |\nabla \text{Ric}_\omega| + \|\Omega\|^2 (1 + |T|) |\text{Ric}_\omega| |\nabla \text{Ric}_\omega| \right. \\
& + (|\text{Ric}_\omega| + |\text{Ric}_\omega|^2) |T|^2 |\nabla T| + |Rm| |\text{Ric}_\omega|^2 + |Rm| |\text{Ric}_\omega| |T|^2 \\
& + |\text{Ric}_\omega|^2 |T|^2 + |\text{Ric}_\omega| |T|^4 + |\text{Ric}_\omega|^3 (|T| + 1)^2 + |\text{Ric}_\omega|^4 \\
& \left. + \|\Omega\|^2 |\text{Ric}_\omega| (|T| + 1)^4 (|\text{Ric}_\omega| + |Rm| + |\nabla T| + 1) \right\}.
\end{aligned}$$

First, we estimate

$$C(|\text{Ric}_\omega| + |\text{Ric}_\omega|^2) |T|^2 |\nabla T| \leq |\text{Ric}_\omega| |\nabla T|^2 + \frac{C^2}{2} |\text{Ric}_\omega| (1 + |\text{Ric}_\omega|)^2 |T|^4. \quad (5.26)$$

$$C|T| |\text{Ric}_\omega| |\nabla \text{Ric}_\omega| \leq \frac{1}{2} |\alpha' \text{Ric}_\omega| |\nabla \text{Ric}_\omega|^2 + \frac{C^2}{2|\alpha'|} |\text{Ric}_\omega| |T|^2, \quad (5.27)$$

We may estimate, using $|T| \leq 1$,

$$C\|\Omega\|^2 |\text{Ric}_\omega| (|T| + 1)^4 |\nabla T| \leq \|\Omega\|^2 |\text{Ric}_\omega| |\nabla T|^2 + \frac{C^2}{4} (2)^8 \|\Omega\|^2 |\text{Ric}_\omega|, \quad (5.28)$$

$$C\|\Omega\|^2 (1 + |T|) |\text{Ric}_\omega| |\nabla \text{Ric}_\omega| \leq \frac{1}{2} |\alpha' \text{Ric}_\omega| |\nabla \text{Ric}_\omega|^2 + \frac{1}{2|\alpha'|} (2C\|\Omega\|^2)^2 |\text{Ric}_\omega|. \quad (5.29)$$

Recall that

$$R_{\bar{k}j}{}^\alpha{}_\beta = \hat{R}_{\bar{k}j}{}^\alpha{}_\beta + \frac{1}{2} R_{\bar{k}j}. \quad (5.30)$$

Hence, using $\|\Omega\| \leq 1$, $|T| \leq 1$ and $|\alpha' \text{Ric}_\omega| \leq 1$ on lower order terms, from (5.25) and the above estimates, we get

$$\begin{aligned}
\partial_t |\text{Ric}_\omega|^2 & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\text{Ric}_\omega|^2 - \left(\frac{1}{2} - 2|\alpha' \text{Ric}_\omega| \right) |\nabla \text{Ric}_\omega|^2 + (5 + C\|\Omega\|^2) |\text{Ric}_\omega| |\nabla T|^2 \right\} \\
& + \frac{C}{2\|\Omega\|} \left\{ |\text{Ric}_\omega| |T|^2 + |\text{Ric}_\omega|^2 + \|\Omega\|^2 |\text{Ric}_\omega| \right\}.
\end{aligned} \quad (5.31)$$

In terms of $0 < \epsilon, \delta < 1$, we have

$$\begin{aligned}
\partial_t |\alpha' \text{Ric}_\omega|^2 & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\alpha' \text{Ric}_\omega|^2 - \left(\frac{1}{2} - 2\epsilon \right) |\alpha' \nabla \text{Ric}_\omega|^2 \right. \\
& \left. + (5 + C\delta^2) \epsilon |\alpha'| |\nabla T|^2 + C\delta \epsilon + C\epsilon^2 \right\}.
\end{aligned} \quad (5.32)$$

Using the evolution of the torsion (4.28)

$$\begin{aligned}
\partial_t |T|^2 & = \frac{1}{2\|\Omega\|} \left\{ \Delta_F |T|^2 - |\nabla T|_{Fg}^2 - \frac{1}{4} |\text{Ric}_\omega|_{Fg}^2 - 2\text{Re}\{g^{i\bar{j}} g^{p\bar{q}} \nabla_i T_p \bar{T}_{\bar{q}} \bar{T}_{\bar{j}}\} \right. \\
& - \text{Re}\{g^{i\bar{j}} g^{p\bar{q}} T_p R_{\bar{q}i} \bar{T}_{\bar{j}}\} - \frac{1}{2} R |T|^2 + \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) |T|^2 \\
& + \text{Re}\{F^{p\bar{q}} g^{i\bar{j}} T^\lambda_{pi} R_{\bar{q}\lambda} \bar{T}_{\bar{j}}\} - F^{p\bar{q}} g^{i\bar{j}} T_i (\hat{R}_{\bar{q}p}{}^{\bar{\lambda}} + R_{\bar{q}p} \delta_j^{\bar{\lambda}}) T_{\bar{\lambda}} + |T|^4 \\
& \left. + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} |T|^2 - |T|^2 \|\Omega\|^2 \nu + 2\text{Re}\langle E, T \rangle \right\}.
\end{aligned} \quad (5.33)$$

Estimating by replacing $F^{p\bar{q}}$ by the evolving metric $g^{p\bar{q}}$,

$$\begin{aligned} \partial_t |T|^2 \leq & \frac{1}{2\|\Omega\|} \left\{ \Delta_F |T|^2 - \frac{1}{2} |\nabla T|^2 - \frac{1}{8} |\text{Ric}_\omega|^2 + 2|\nabla T||T|^2 + C|\text{Ric}_\omega||T|^2 \right. \\ & + |R||T|^2 + \frac{|\alpha'|}{4} |\text{Ric}_\omega|^2 |T|^2 + |\hat{R}m||T|^2 + |T|^4 \\ & \left. + C\|\Omega\|^2(|T|^4 + |T|^3 + |T|^2 + |T|)(1 + |\text{Ric}_\omega| + |\nabla T|) \right\}. \end{aligned} \quad (5.34)$$

Estimate

$$2|\nabla T||T|^2 \leq \frac{1}{8} |\nabla T|^2 + 8|T|^4, \quad (5.35)$$

and

$$C\|\Omega\|^2(|T|^4 + |T|^3 + |T|^2 + |T|)|\nabla T| \leq \frac{1}{8} |\nabla T|^2 + 2C^2\|\Omega\|^4(4)^2. \quad (5.36)$$

Using $0 < \delta, \epsilon < 1$,

$$\partial_t |T|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |T|^2 - \frac{1}{4} |\nabla T|^2 - \frac{1}{8} |\text{Ric}_\omega|^2 + C\epsilon\delta + C\delta \right\}. \quad (5.37)$$

Combining (5.32) and (5.37), we obtain the desired estimate.

Theorem 5 *Start the flow with a constant function $u_0 = \log M$. There exists $M_0 \gg 1$ such that for every $M \geq M_0$, if*

$$\|\Omega\|^2 \leq \frac{C_2^2}{M^2}, \quad |T|^2 \leq \frac{C_3}{M^{4/3}}, \quad (5.38)$$

along the flow, then

$$|\alpha' \text{Ric}_\omega| \leq \frac{1}{M^{1/2}}. \quad (5.39)$$

Here, C_2 and C_3 are the constants given in Theorems 3 and 4 respectively.

Proof: Denote

$$\epsilon = \frac{1}{M^{1/2}}, \quad \delta = \frac{C_3}{M^{4/3}}. \quad (5.40)$$

Let C_4 denote the largest of the constants C on the right-hand side of (5.22). For M_0 large enough, we can simultaneously satisfy the hypothesis of Lemma 1, and the inequalities $2\epsilon < \frac{1}{2}$ and $(5 + C_4\delta^2)\epsilon \leq 1$. We will study the evolution equation of

$$|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2, \quad (5.41)$$

where Λ is a constant given by

$$\Lambda = \max\{4|\alpha'|, 16|\alpha'|^2(C_4 + 1)\}. \quad (5.42)$$

With this choice of Λ and M_0 , we have

$$\left(\frac{1}{2} - 2\epsilon\right) \geq 0, \quad \left(\frac{\Lambda}{4} - (5 + C_4\delta^2)\epsilon|\alpha'|\right) \geq 0. \quad (5.43)$$

At $t = 0$, $u_0 = \log M$ and it follows that

$$\alpha'^2 |\text{Ric}_\omega|^2 + \Lambda |T|^2 = 0. \quad (5.44)$$

Suppose that along the flow, we reach

$$\alpha'^2 |\text{Ric}_\omega|^2 + \Lambda |T|^2 = \frac{\epsilon^2}{2}, \quad (5.45)$$

at some point $p \in X$ at a first time $t_0 > 0$. By Lemma 1,

$$\partial_t(|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2) \leq \frac{1}{2\|\Omega\|} \left\{ -\frac{\Lambda}{8} |\text{Ric}_\omega|^2 + C_4(1 + \Lambda)\epsilon\delta + C_4\epsilon^2 + C_4\Lambda\delta \right\}. \quad (5.46)$$

At (p, t_0) , we have

$$|\alpha' \text{Ric}_\omega|^2 = \frac{\epsilon^2}{2} - \Lambda |T|^2 \geq \frac{\epsilon^2}{2} - \Lambda\delta. \quad (5.47)$$

Thus

$$\partial_t(|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2) \leq \frac{1}{2\|\Omega\|} \left\{ -\frac{\Lambda}{16|\alpha'|^2} \epsilon^2 + C_4\epsilon^2 + \frac{\Lambda^2\delta}{8|\alpha'|^2} + C_4\Lambda\delta + C_4(1 + \Lambda)\epsilon\delta \right\}.$$

By our choice of Λ (5.42), and after substituting the definition of ϵ and δ , we obtain

$$\partial_t(|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2) \leq \frac{1}{2\|\Omega\|} \left\{ -\frac{1}{M} + \left(\frac{\Lambda^2}{8|\alpha'|^2} + C\Lambda \right) \frac{1}{M^{4/3}} + C(1 + \Lambda) \frac{1}{M^{1/2}} \frac{1}{M^{4/3}} \right\}.$$

Hence for $M_0 \gg 1$ depending only on $(X, \hat{\omega})$, α' , μ , ρ , for all $M \geq M_0$ we have at (p, t_0)

$$\partial_t(|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2) \leq 0. \quad (5.48)$$

Hence along the flow, there holds

$$|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2 \leq \frac{\epsilon^2}{2}. \quad (5.49)$$

It follows that

$$|\alpha' \text{Ric}_\omega| < \epsilon \quad (5.50)$$

is preserved along the flow.

6 Higher order estimates

6.1 The evolution of derivatives of torsion

6.1.1 Covariant derivative of torsion

Since $\nabla_{\bar{k}} T_j = \frac{1}{2} R_{\bar{k}j}$, we only need to look at $\nabla_k T_j$. We will compute

$$\partial_t \nabla_i T_j = \nabla_i \partial_t T_j - \partial_t \Gamma_{ij}^\lambda T_\lambda. \quad (6.1)$$

First, using the standard formula for the evolution of the Christoffel symbols and (1.1), we compute

$$\begin{aligned} \partial_t \Gamma_{ij}^\lambda &= g^{\lambda\bar{\mu}} \nabla_i \partial_t g_{\bar{\mu}j} \\ &= \nabla_i \left\{ \frac{1}{2\|\Omega\|} \left(-\frac{R}{2} - \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + |T|^2 + \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) + \alpha' \|\Omega\|^2 \nu \right) \right\} \delta^\lambda_j \\ &= \frac{1}{2\|\Omega\|} \left\{ -\frac{1}{2} \nabla_i R - \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_i R_{\bar{q}p} + \frac{\alpha'}{4} \sigma_2^{p\bar{q}} \nabla_i R_{\bar{q}p} + g^{p\bar{q}} \nabla_i T_p \bar{T}_{\bar{q}} \right. \\ &\quad \left. + \frac{1}{2} g^{p\bar{q}} T_p R_{\bar{q}i} + \frac{R}{2} T_i - |T|^2 T_i - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) T_i - E_i \right\} \delta^\lambda_j. \end{aligned} \quad (6.2)$$

We recall that the definition of E_i is given in (4.14). Using (4.20)

$$\begin{aligned} \partial_t \nabla_i T_j &= \frac{1}{2\|\Omega\|} \nabla_i \left\{ F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j - \nabla_j |T|^2 - \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2(i\text{Ric}_\omega) + \frac{1}{2} F^{p\bar{q}} T^\lambda_{pj} R_{\bar{q}\lambda} \right. \\ &\quad \left. + T_j |T|^2 + E_j \right\} + \nabla_i \left\{ \frac{1}{2\|\Omega\|} \right\} \left\{ F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j - g^{p\bar{q}} \nabla_j T_p \bar{T}_{\bar{q}} - \frac{1}{2} g^{p\bar{q}} T_p R_{\bar{q}j} \right. \\ &\quad \left. - \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2(i\text{Ric}_\omega) + \frac{1}{2} F^{p\bar{q}} T^\lambda_{pj} R_{\bar{q}\lambda} + T_j |T|^2 + E_j \right\} \\ &\quad - \frac{1}{2\|\Omega\|} \left\{ -F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_i + g^{p\bar{q}} \nabla_i T_p \bar{T}_{\bar{q}} + \frac{1}{2} g^{p\bar{q}} T_p R_{\bar{q}i} \right. \\ &\quad \left. + \frac{R}{2} T_i - |T|^2 T_i - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) T_i - E_i \right\} T_j. \end{aligned} \quad (6.3)$$

First, we may rewrite

$$F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j = \frac{1}{2} F^{p\bar{q}} \nabla_p R_{\bar{q}j}. \quad (6.4)$$

Next,

$$\begin{aligned} \nabla_i \{ F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j \} &= F^{p\bar{q}} \nabla_i \nabla_p \nabla_{\bar{q}} T_j + \nabla_i \left(\alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}} \right) \nabla_p \nabla_{\bar{q}} T_j \\ &= F^{p\bar{q}} \nabla_p \nabla_i \nabla_{\bar{q}} T_j + F^{p\bar{q}} T^\lambda_{pi} \nabla_\lambda \nabla_{\bar{q}} T_j + \alpha' \nabla_i (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_p \nabla_{\bar{q}} T_j \\ &\quad - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p \nabla_{\bar{q}} T_j \\ &= F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} \nabla_i T_j - F^{p\bar{q}} \nabla_p (R_{\bar{q}i}^\lambda T_\lambda) + F^{p\bar{q}} T^\lambda_{pi} \nabla_\lambda R_{\bar{q}j} \\ &\quad + \frac{\alpha'}{2} \nabla_i (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_p R_{\bar{q}j} - \frac{\alpha'}{4} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p R_{\bar{q}j}. \end{aligned} \quad (6.5)$$

We also compute

$$\begin{aligned} \nabla_i \nabla_j |T|^2 &= g^{p\bar{q}} \nabla_i \nabla_j T_p \bar{T}_{\bar{q}} + g^{p\bar{q}} \nabla_j T_p \nabla_i \bar{T}_{\bar{q}} + g^{p\bar{q}} \nabla_i T_p \nabla_j \bar{T}_{\bar{q}} + g^{p\bar{q}} T_p \nabla_i \nabla_j \bar{T}_{\bar{q}} \\ &= g^{p\bar{q}} \nabla_i \nabla_j T_p \bar{T}_{\bar{q}} + \frac{1}{2} g^{p\bar{q}} \nabla_j T_p R_{\bar{q}i} + \frac{1}{2} g^{p\bar{q}} \nabla_i T_p R_{\bar{q}j} + \frac{1}{2} g^{p\bar{q}} T_p \nabla_i R_{\bar{q}j}. \end{aligned} \quad (6.6)$$

We introduce the notation \mathcal{E} , which denotes any combination of terms involving only Rm , T , g , $\|\Omega\|$, α' , ρ and μ , as well as any derivatives of ρ and μ . Note that $F^{p\bar{q}}$ is an element of \mathcal{E} . The notation $*$ refers to a contraction using the evolving metric g . The notation $D\mathcal{E}$ denotes any term which is a covariant derivative of a term in \mathcal{E} . For example, the group $D\mathcal{E}$ contains terms involving ∇T , $\bar{\nabla} \bar{T}$, and ∇Ric_ω . Substituting (6.4), (6.5), (6.6) gives

$$\partial_t \nabla_i T_j = \frac{1}{2\|\Omega\|} \left\{ \Delta_F \nabla_i T_j - \frac{\alpha'}{4} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p R_{\bar{q}j} + \nabla \nabla T * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E} \right\}. \quad (6.7)$$

Here we also used that $\nabla_i E_j = \nabla \nabla T * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E}$ which can be verified from the definition of E_j given in (4.14)

6.1.2 Norm of covariant derivative of torsion

We will compute

$$\partial_t |\nabla T|^2 = \partial_t \{ g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i T_k \nabla_{\bar{j}} \bar{T}_{\bar{\ell}} \}. \quad (6.8)$$

As in (4.25), we have

$$\begin{aligned} \partial_t |\nabla T|^2 &= 2\text{Re} \langle \partial_t \nabla T, \nabla T \rangle \\ &\quad + 2 \frac{|\nabla T|^2}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right). \end{aligned} \quad (6.9)$$

Next,

$$\begin{aligned} \Delta_F |\nabla T|^2 &= F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_p \nabla_{\bar{q}} \nabla_i T_k \nabla_{\bar{j}} \bar{T}_{\bar{\ell}} + g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i T_k \overline{F^{p\bar{q}} \nabla_{\bar{p}} \nabla_q \nabla_j \bar{T}_{\bar{\ell}}} \\ &\quad + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_p \nabla_i T_k \nabla_{\bar{q}} \nabla_{\bar{j}} \bar{T}_{\bar{\ell}} + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_{\bar{q}} \nabla_i T_k \nabla_p \nabla_{\bar{j}} \bar{T}_{\bar{\ell}} \\ &= 2\text{Re} \langle \Delta_F \nabla T, \nabla T \rangle + g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i T_k F^{p\bar{q}} R_{\bar{q}p\bar{j}}^{\bar{\lambda}} \nabla_{\bar{\lambda}} \bar{T}_{\bar{\ell}} \\ &\quad + g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i T_k F^{p\bar{q}} R_{\bar{q}p\bar{\ell}}^{\bar{\lambda}} \nabla_{\bar{j}} \bar{T}_{\bar{\lambda}} + |\nabla \nabla T|_{Fg}^2 + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_{\bar{q}} \nabla_i T_k \nabla_p \nabla_{\bar{j}} \bar{T}_{\bar{\ell}}. \end{aligned}$$

The last term can be written as a norm of ∇Ric_ω plus commutator terms. Explicitly,

$$\begin{aligned} F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_{\bar{q}} \nabla_i T_k \nabla_p \nabla_{\bar{j}} \bar{T}_{\bar{\ell}} &= F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i \nabla_{\bar{q}} T_k \overline{\nabla_{\bar{p}} \nabla_j \bar{T}_{\bar{\ell}}} + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} \nabla_p \nabla_{\bar{j}} \bar{T}_{\bar{\ell}} \\ &= F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i \nabla_{\bar{q}} T_k \overline{\nabla_j \nabla_{\bar{p}} \bar{T}_{\bar{\ell}}} + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i \nabla_{\bar{q}} T_k R_{\bar{j}p\bar{\ell}}^{\bar{\lambda}} T_{\bar{\lambda}} \\ &\quad + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} \nabla_{\bar{j}} \nabla_p \bar{T}_{\bar{\ell}} + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} R_{\bar{j}p\bar{\ell}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} \\ &= \frac{1}{4} F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i R_{\bar{q}k} \overline{\nabla_j \nabla_{\bar{p}} \bar{T}_{\bar{\ell}}} + \frac{1}{2} F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i R_{\bar{q}k} R_{\bar{j}p\bar{\ell}}^{\bar{\lambda}} T_{\bar{\lambda}} \\ &\quad + \frac{1}{2} F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} \nabla_{\bar{j}} R_{\bar{\ell}p} + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} R_{\bar{j}p\bar{\ell}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}}. \end{aligned} \quad (6.10)$$

Hence

$$\begin{aligned}
\Delta_F |\nabla T|^2 &= 2\operatorname{Re}\langle \Delta_F \nabla T, \nabla T \rangle + |\nabla \nabla T|_{Fgg}^2 + \frac{1}{4} |\nabla \operatorname{Ric}_\omega|_{Fgg}^2 \\
&\quad + g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i T_k F^{p\bar{q}} R_{\bar{q}p\bar{j}}^{\bar{\lambda}} \nabla_{\bar{\lambda}} \bar{T}_{\bar{\ell}} + g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i T_k F^{p\bar{q}} R_{\bar{q}p\bar{\ell}}^{\bar{\lambda}} \nabla_{\bar{j}} \bar{T}_{\bar{\lambda}} \\
&\quad + \frac{1}{2} F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} \nabla_i R_{\bar{q}k} R_{\bar{j}p\bar{\ell}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} + \frac{1}{2} F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} \nabla_{\bar{j}} R_{\bar{\ell}p} \\
&\quad + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} R_{\bar{q}i}^{\lambda} T_{\lambda} R_{\bar{j}p\bar{\ell}}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}}.
\end{aligned} \tag{6.11}$$

Therefore, by (6.7), (6.9) and (6.11),

$$\begin{aligned}
\partial_t |\nabla T|^2 &= \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\nabla T|^2 - |\nabla \nabla T|_{Fgg}^2 - \frac{1}{4} |\nabla \operatorname{Ric}_\omega|_{Fgg}^2 \right. \\
&\quad \left. - \frac{\alpha'}{2} \operatorname{Re}\{g^{i\bar{j}} g^{k\bar{\ell}} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p R_{\bar{q}k} \nabla_{\bar{j}} \bar{T}_{\bar{\ell}}\} + \nabla \nabla T * \nabla T * \mathcal{E} \right. \\
&\quad \left. + D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E} \right\}.
\end{aligned} \tag{6.12}$$

6.2 The evolution of derivatives of curvature

6.2.1 Derivative of Ricci curvature

We will compute

$$\partial_t \nabla_i R_{\bar{k}j} = \nabla_i \partial_t R_{\bar{k}j} - \partial_t \Gamma_{ij}^{\lambda} R_{\bar{k}\lambda}. \tag{6.13}$$

Using (5.13) and (6.2), we obtain

$$\begin{aligned}
\partial_t \nabla_i R_{\bar{k}j} &= \frac{1}{2\|\Omega\|} \left\{ \nabla_i (F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j}) - \frac{\alpha'}{2} \nabla_i (\sigma_2^{p\bar{q}, r\bar{s}} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p}) \right. \\
&\quad \left. + (2g^{p\bar{q}} + 2\alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) * \nabla \nabla T * \nabla T + DD\mathcal{E} * \mathcal{E} \right. \\
&\quad \left. + D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E} \right\}.
\end{aligned} \tag{6.14}$$

Here, we used that $\nabla \bar{\nabla} \bar{T} = \bar{\nabla} \operatorname{Ric}_\omega + Rm * \bar{T}$. Compute

$$\begin{aligned}
\nabla_i (F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j}) &= F^{p\bar{q}} \nabla_i \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} + \alpha' \nabla_i (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} - \frac{\alpha'}{2} \nabla_i (\sigma_2^{p\bar{q}}) \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} \\
&= F^{p\bar{q}} \nabla_p \nabla_i \nabla_{\bar{q}} R_{\bar{k}j} + F^{p\bar{q}} T_{pi}^{\lambda} \nabla_{\lambda} \nabla_{\bar{q}} R_{\bar{k}j} \\
&\quad + \alpha' \nabla_i (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} \\
&= F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} \nabla_i R_{\bar{k}j} + F^{p\bar{q}} \nabla_p (R_{\bar{q}i\bar{k}}^{\bar{\lambda}} R_{\bar{\lambda}j} - R_{\bar{q}i}^{\lambda} R_{\bar{\lambda}j}) \\
&\quad + F^{p\bar{q}} T_{pi}^{\lambda} \nabla_{\lambda} \nabla_{\bar{q}} R_{\bar{k}j} + \alpha' \nabla_i (\|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} \\
&\quad - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j}.
\end{aligned} \tag{6.15}$$

Hence, using that $\nabla_i \sigma_2^{p\bar{q}, r\bar{s}} = 0$ (5.10), we obtain

$$\begin{aligned} \partial_t \nabla_i R_{\bar{k}j} &= \frac{1}{2\|\Omega\|} \left\{ \Delta_F \nabla_i R_{\bar{k}j} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p} \right. \\ &\quad - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_i \nabla_j R_{\bar{q}p} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} \\ &\quad + (2g^{p\bar{q}} + 2\alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}}) * \nabla \nabla T * \nabla T \\ &\quad \left. + DD\mathcal{E} * \mathcal{E} + D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E} \right\}. \end{aligned} \quad (6.16)$$

6.2.2 Norm of derivative of Ricci curvature

We will compute

$$\partial_t |\nabla \text{Ric}_\omega|^2 = \partial_t \{ g^{i\bar{a}} g^{b\bar{k}} g^{j\bar{c}} \nabla_i R_{\bar{k}j} \overline{\nabla_a R_{\bar{b}c}} \}. \quad (6.17)$$

As in (4.25), we have

$$\begin{aligned} \partial_t |\nabla \text{Ric}_\omega|^2 &= 2\text{Re} \langle \partial_t \nabla \text{Ric}_\omega, \nabla \text{Ric}_\omega \rangle \\ &\quad + 3 |\nabla \text{Ric}_\omega|^2 \frac{1}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right). \end{aligned}$$

Next, compute

$$\begin{aligned} &\Delta_F |\nabla \text{Ric}_\omega|^2 \\ &= F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} \nabla_p \nabla_{\bar{q}} \nabla_i R_{\bar{n}k} \overline{\nabla_j R_{\bar{m}\ell}} \\ &\quad + g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} \nabla_i R_{\bar{n}k} \overline{F^{q\bar{p}} \nabla_{\bar{p}} \nabla_q \nabla_j R_{\bar{m}\ell}} + |\nabla \nabla \text{Ric}_\omega|_{Fggg}^2 + |\overline{\nabla} \nabla \text{Ric}_\omega|_{Fggg}^2 \\ &= 2\text{Re} \langle \Delta_F \nabla \text{Ric}_\omega, \nabla \text{Ric}_\omega \rangle + |\nabla \nabla \text{Ric}_\omega|_{Fggg}^2 + |\overline{\nabla} \nabla \text{Ric}_\omega|_{Fggg}^2 \\ &\quad + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} \nabla_i R_{\bar{n}k} R_{\bar{q}p\bar{j}}^{\bar{\lambda}} \nabla_{\bar{\lambda}} R_{\bar{\ell}m} + F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} \nabla_i R_{\bar{n}k} R_{\bar{q}p\bar{\ell}}^{\bar{\lambda}} \nabla_{\bar{j}} R_{\bar{\lambda}m} \\ &\quad - F^{p\bar{q}} g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} \nabla_i R_{\bar{n}k} R_{\bar{q}p}^{\bar{\lambda}} \nabla_{\bar{j}} R_{\bar{\ell}\bar{\lambda}}. \end{aligned} \quad (6.18)$$

Commuting covariant derivatives

$$|\overline{\nabla} \nabla \text{Ric}_\omega|_{Fggg}^2 = |\nabla \overline{\nabla} \text{Ric}_\omega|_{Fggg}^2 + \nabla \overline{\nabla} \mathcal{E} * \mathcal{E} + \mathcal{E}. \quad (6.19)$$

Hence

$$\begin{aligned} \partial_t |\nabla \text{Ric}_\omega|^2 &= \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\nabla \text{Ric}_\omega|^2 - |\nabla \nabla \text{Ric}_\omega|_{Fggg}^2 - |\nabla \overline{\nabla} \text{Ric}_\omega|_{Fggg}^2 \right\} \\ &\quad + \frac{1}{2\|\Omega\|} 2\text{Re} \left\{ - \frac{\alpha'}{2} g^{i\bar{a}} g^{b\bar{k}} g^{j\bar{c}} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p} \overline{\nabla_a R_{\bar{b}c}} \right. \\ &\quad - \frac{\alpha'}{2} g^{i\bar{a}} g^{b\bar{k}} g^{j\bar{c}} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_i \nabla_j R_{\bar{q}p} \overline{\nabla_a R_{\bar{b}c}} \\ &\quad \left. - \frac{\alpha'}{2} g^{i\bar{a}} g^{b\bar{k}} g^{j\bar{c}} \sigma_2^{p\bar{q}, r\bar{s}} \nabla_i R_{\bar{s}r} \nabla_p \nabla_{\bar{q}} R_{\bar{k}j} \overline{\nabla_a R_{\bar{b}c}} \right\} \end{aligned}$$

$$\begin{aligned}
& +(2g^{p\bar{q}} + 2\alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}) * \nabla\nabla T * \nabla T * \nabla\text{Ric}_\omega \Big\} \\
& + DDE * D\mathcal{E} * \mathcal{E} + DDE * \mathcal{E} + D\mathcal{E} * D\mathcal{E} * D\mathcal{E} * \mathcal{E} \\
& + D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E}.
\end{aligned} \tag{6.20}$$

Lemma 2 Suppose $|\alpha'\text{Ric}_\omega| \leq \frac{1}{4}$ and $-\frac{1}{8}g^{p\bar{q}} < \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}} < \frac{1}{8}g^{p\bar{q}}$. Then

$$\begin{aligned}
\partial_t |\nabla\text{Ric}_\omega|^2 & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\nabla\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla\nabla\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla\bar{\nabla}\text{Ric}_\omega|^2 \right\} \\
& + \frac{1}{2\|\Omega\|} \left\{ 9\alpha'^2 |\nabla\text{Ric}_\omega|^4 + 5 |\nabla\nabla T| |\nabla T| |\nabla\text{Ric}_\omega| \right. \\
& \left. + DDE * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \right\}.
\end{aligned} \tag{6.21}$$

Proof: By assumption, we may use

$$|\nabla\nabla\text{Ric}_\omega|_{F_{ggg}}^2 + |\nabla\bar{\nabla}\text{Ric}_\omega|_{F_{ggg}}^2 \geq \frac{3}{4} (|\nabla\nabla\text{Ric}_\omega|^2 + |\nabla\bar{\nabla}\text{Ric}_\omega|^2). \tag{6.22}$$

In coordinates where the evolving metric g is the identity, we have $\sigma_2^{p\bar{q}, r\bar{s}} = \pm 1$. Using $2ab \leq a^2 + b^2$, estimate (6.21) follows from (6.20).

6.3 Higher order estimates

Theorem 6 There exists $0 < \delta_1, \delta_2$ with the following property. Suppose

$$-\frac{1}{8}g^{p\bar{q}} < \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}} < \frac{1}{8}g^{p\bar{q}}, \quad \|\Omega\| \leq 1, \tag{6.23}$$

$$|\alpha'\text{Ric}_\omega| \leq \delta_1, \tag{6.24}$$

and

$$|T|^2 \leq \delta_2, \tag{6.25}$$

along the flow. Then

$$|\nabla\text{Ric}_\omega| \leq C, \quad |\nabla T| \leq C, \tag{6.26}$$

where C depends only on $\delta_1, \delta_2, \alpha', \rho, \mu$, and $(X, \hat{\omega})$.

Proof: Let us assume that $\delta_1 < \frac{1}{4}$. This will allow us to use the estimate

$$\frac{3}{4}g_{\bar{k}j} \leq F^{j\bar{k}} \leq 2g_{\bar{k}j}. \tag{6.27}$$

This follows from the definition of $F^{j\bar{k}}$, see (4.22). From (5.20), with assumptions (6.24) and (6.27) we may estimate

$$\begin{aligned}
\partial_t |\text{Ric}_\omega|^2 & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla\text{Ric}_\omega|^2 \right\} \\
& + \frac{1}{2\|\Omega\|} \text{Re} \left\{ D\mathcal{E} * \mathcal{E} + 5 \nabla T * \nabla T * \text{Ric} + \mathcal{E} \right\}.
\end{aligned} \tag{6.28}$$

Here we used

$$-\alpha' \text{Re}\{g^{j\bar{\ell}} g^{m\bar{k}} \sigma_2^{p\bar{q}, r\bar{s}} R_{\bar{\ell}m} \nabla_{\bar{k}} R_{\bar{s}r} \nabla_j R_{\bar{q}p}\} \leq \delta_1 |\nabla \text{Ric}_\omega|^2, \quad (6.29)$$

to absorb this term into the $-|\nabla \text{Ric}_\omega|^2$ term. We will compute the evolution of

$$G = (|\alpha' \text{Ric}_\omega|^2 + \tau_1) |\nabla \text{Ric}_\omega|^2 + (|T|^2 + \tau_2) |\nabla T|^2, \quad (6.30)$$

where τ_1 and τ_2 are constants to be determined. First, we compute

$$\partial_t \{(|\alpha' \text{Ric}_\omega|^2 + \tau_1) |\nabla \text{Ric}_\omega|^2\} = \alpha'^2 \partial_t |\text{Ric}_\omega|^2 |\nabla \text{Ric}_\omega|^2 + (|\alpha' \text{Ric}_\omega|^2 + \tau_1) \partial_t |\nabla \text{Ric}_\omega|^2. \quad (6.31)$$

By (6.21) and (6.28)

$$\begin{aligned} & \partial_t \{(|\alpha' \text{Ric}_\omega|^2 + \tau_1) |\nabla \text{Ric}_\omega|^2\} \\ & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\alpha' \text{Ric}_\omega|^2 |\nabla \text{Ric}_\omega|^2 - \frac{\alpha'^2}{2} |\nabla \text{Ric}_\omega|^4 \right\} \\ & \quad + \frac{1}{2\|\Omega\|} \text{Re} \left\{ D\mathcal{E} * \mathcal{E} + 5 \nabla T * \nabla T * \text{Ric} + \mathcal{E} \right\} \alpha'^2 |\nabla \text{Ric}_\omega|^2 \\ & \quad + \frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|} \left\{ \Delta_F |\nabla \text{Ric}_\omega|^2 - \frac{1}{2} |\nabla \nabla \text{Ric}_\omega|^2 - \frac{1}{2} |\nabla \bar{\nabla} \text{Ric}_\omega|^2 \right\} \\ & \quad + \frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|} \left\{ 9\alpha'^2 |\nabla \text{Ric}_\omega|^4 + 5 |\nabla \nabla T| |\nabla T| |\nabla \text{Ric}_\omega| \right. \\ & \quad \left. + \nabla \nabla \mathcal{E} * D\mathcal{E} * \mathcal{E} + \nabla \bar{\nabla} \mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \right\}. \end{aligned} \quad (6.32)$$

Hence

$$\begin{aligned} & \partial_t \{(|\alpha' \text{Ric}_\omega|^2 + \tau_1) |\nabla \text{Ric}_\omega|^2\} \\ & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F \{(|\alpha' \text{Ric}_\omega|^2 + \tau_1) |\nabla \text{Ric}_\omega|^2\} - \left(\frac{1}{2} - 9|\alpha' \text{Ric}_\omega|^2 - 9\tau_1 \right) \alpha'^2 |\nabla \text{Ric}_\omega|^4 \right. \\ & \quad - \frac{1}{2} |\nabla \nabla \text{Ric}_\omega|^2 (|\alpha' \text{Ric}_\omega|^2 + \tau_1) - \frac{1}{2} |\nabla \bar{\nabla} \text{Ric}_\omega|^2 (|\alpha' \text{Ric}_\omega|^2 + \tau_1) \\ & \quad \left. - 2 \text{Re} \{ F^{i\bar{j}} \nabla_i |\alpha' \text{Ric}_\omega|^2 \nabla_{\bar{j}} |\nabla \text{Ric}_\omega|^2 \} + 6(\delta_1^2 + \tau) |\nabla \nabla T| |\nabla T| |\nabla \text{Ric}_\omega| \right\} \\ & \quad + \frac{\alpha'^2 |\nabla \text{Ric}_\omega|^2}{2\|\Omega\|} \text{Re} \left\{ 5 \nabla T * \nabla T * \text{Ric} + D\mathcal{E} * \mathcal{E} + \mathcal{E} \right\} \\ & \quad + \frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|} \left\{ \nabla \nabla \mathcal{E} * D\mathcal{E} * \mathcal{E} + \nabla \bar{\nabla} \mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \right\}. \end{aligned} \quad (6.33)$$

We estimate

$$\begin{aligned} & -2 \text{Re} \{ F^{i\bar{j}} \nabla_i |\alpha' \text{Ric}_\omega|^2 \nabla_{\bar{j}} |\nabla \text{Ric}_\omega|^2 \} \\ & \leq 8|\alpha'| \delta_1 |\nabla \text{Ric}_\omega|^2 (|\nabla \nabla \text{Ric}_\omega| + |\nabla \bar{\nabla} \text{Ric}_\omega| + \mathcal{E}) \\ & \leq \frac{\alpha'^2}{2^4} |\nabla \text{Ric}_\omega|^4 + 2^8 \delta_1^2 (|\nabla \nabla \text{Ric}_\omega|^2 + |\nabla \bar{\nabla} \text{Ric}_\omega|^2) + C |\nabla \text{Ric}_\omega|^2, \end{aligned} \quad (6.34)$$

$$\begin{aligned}
& 6(\delta_1^2 + \tau_1)|\nabla\nabla T||\nabla T||\nabla\text{Ric}_\omega| \\
& \leq \frac{1}{2}(\delta_1^2 + \tau_1)|\nabla\nabla T|^2 + 2^13^2(\delta_1^2 + \tau_1)|\nabla T|^2|\nabla\text{Ric}_\omega|^2 \\
& \leq \frac{1}{2}(\delta_1^2 + \tau_1)|\nabla\nabla T|^2 + \frac{\alpha'^2}{2^4}|\nabla\text{Ric}_\omega|^4 + 2^43^4\alpha'^{-2}(\delta_1^2 + \tau_1)^2|\nabla T|^4, \tag{6.35}
\end{aligned}$$

$$\begin{aligned}
& \frac{\alpha'^2|\nabla\text{Ric}_\omega|^2}{2\|\Omega\|}\text{Re}\left\{5\nabla T * \nabla T * \text{Ric} + \nabla\mathcal{E} * \mathcal{E} + \mathcal{E}\right\} \\
& \leq \frac{1}{2\|\Omega\|}\left\{\frac{\alpha'^2}{2^4}|\nabla\text{Ric}_\omega|^4 + 2^25^2\delta_1^2|\nabla T|^4 + C|\nabla\text{Ric}_\omega|^3 + C|\nabla T|^3 + C\right\}. \tag{6.36}
\end{aligned}$$

$$\begin{aligned}
& \frac{(|\alpha'\text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|}\left\{\nabla\nabla\mathcal{E} * D\mathcal{E} * \mathcal{E} + \nabla\bar{\nabla}\mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3\right\} \\
& \leq \frac{1}{2\|\Omega\|}\left\{\frac{1}{4}|\nabla\nabla\text{Ric}_\omega|^2(|\alpha'\text{Ric}_\omega|^2 + \tau_1) + \frac{1}{4}|\nabla\bar{\nabla}\text{Ric}_\omega|^2(|\alpha'\text{Ric}_\omega|^2 + \tau_1) \right. \\
& \quad \left. + \frac{1}{2}(\delta_1^2 + \tau_1)|\nabla\nabla T|^2 + C|\nabla\text{Ric}_\omega|^3 + C|\nabla T|^3 + C\right\}. \tag{6.37}
\end{aligned}$$

Therefore

$$\begin{aligned}
& \partial_t\{(|\alpha'\text{Ric}_\omega|^2 + \tau_1)|\nabla\text{Ric}_\omega|^2\} \tag{6.38} \\
& \leq \frac{1}{2\|\Omega\|}\left\{\Delta_F\{(|\alpha'\text{Ric}_\omega|^2 + \tau_1)|\nabla\text{Ric}_\omega|^2\} - \left(\frac{1}{4} - 9\delta_1^2 - 9\tau_1\right)\alpha'^2|\nabla\text{Ric}_\omega|^4 \right. \\
& \quad - (|\nabla\nabla\text{Ric}_\omega|^2 + |\nabla\bar{\nabla}\text{Ric}_\omega|^2)\left(\frac{\tau_1}{4} - 2^8\delta_1^2\right) + (\delta_1^2 + \tau_1)|\nabla\nabla T|^2 \\
& \quad \left. + \left(2^43^4\alpha'^{-2}(\delta_1^2 + \tau_1)^2 + 2^25^2\delta_1^2\right)|\nabla T|^4 + C_{\alpha',\tau,\delta}|\nabla\text{Ric}_\omega|^3 + C_{\alpha',\tau,\delta}|\nabla T|^3 + C_{\alpha',\tau,\delta}\right\}.
\end{aligned}$$

Next, we compute

$$\partial_t\{(|T|^2 + \tau_2)|\nabla T|^2\} = \partial_t|T|^2|\nabla T|^2 + (|T|^2 + \tau_2)\partial_t|\nabla T|^2. \tag{6.39}$$

By (4.28), we have

$$\partial_t|T|^2 \leq \frac{1}{2\|\Omega\|}\left\{\Delta_F|T|^2 - |\nabla T|_{Fg}^2 + C|\nabla T| + C\right\}. \tag{6.40}$$

By (6.12), we have

$$\begin{aligned}
\partial_t|\nabla T|^2 & \leq \frac{1}{2\|\Omega\|}\left\{\Delta_F|\nabla T|^2 - |\nabla\nabla T|_{Fgg}^2 + |\alpha'||\nabla T||\nabla\text{Ric}_\omega|^2 \right. \\
& \quad \left. + C|\nabla\nabla T||\nabla T| + C|\nabla T|^2 + C|\nabla\text{Ric}_\omega|^2 + C\right\}. \tag{6.41}
\end{aligned}$$

By our assumption $|\alpha' \text{Ric}_\omega| \leq \frac{1}{4}$, we have $|\nabla \nabla T|_{Fg}^2 \geq \frac{1}{2} |\nabla \nabla T|^2$ and $|\nabla T|_{Fg}^2 \geq \frac{1}{2} |\nabla T|^2$. Therefore

$$\begin{aligned} & \partial_t \{(|T|^2 + \tau_2) |\nabla T|^2\} \\ & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F \{(|T|^2 + \tau_2) |\nabla T|^2\} - 2\text{Re}\{F^{i\bar{j}} \nabla_i |T|^2 \nabla_{\bar{j}} |\nabla T|^2\} \right. \\ & \quad \left. - \frac{1}{4} |\nabla T|^4 - (|T|^2 + \tau_2) \frac{1}{4} |\nabla \nabla T|^2 + C |\nabla \text{Ric}_\omega|^3 + C |\nabla T|^3 + C \right\}. \end{aligned} \quad (6.42)$$

Here we used Young's inequality $|\nabla T| |\nabla \text{Ric}_\omega|^2 \leq \frac{1}{3} |\nabla T|^3 + \frac{2}{3} |\nabla \text{Ric}_\omega|^3$. In the following, we will use that $\bar{\nabla} T$ can be expressed as Ricci curvature. We estimate

$$\begin{aligned} & -2\text{Re}\{F^{i\bar{j}} \nabla_i |T|^2 \nabla_{\bar{j}} |\nabla T|^2\} \\ & \leq 4|T| |\nabla T| (|\nabla T| + |\bar{\nabla} T|) (|\nabla \nabla T| + |\bar{\nabla} \nabla T|) \\ & \leq 4|T| |\nabla T|^2 |\nabla \nabla T| + 4|T| |\nabla T|^2 |\nabla \text{Ric}_\omega| + 4|T| |\nabla T| |\text{Ric}_\omega| |\nabla \nabla T| \\ & \quad + 4|T| |\nabla T| |\text{Ric}_\omega| |\nabla \text{Ric}_\omega| + 4|T| |\nabla T| (|\nabla T| + |\bar{\nabla} T|) |R * T|. \end{aligned} \quad (6.43)$$

We may estimate the first term in the following way

$$4|T| |\nabla T|^2 |\nabla \nabla T| \leq 4|\nabla T|^2 (\delta_2)^{1/2} |\nabla \nabla T| \leq \frac{1}{2^3} |\nabla T|^4 + 2^5 \delta_2 |\nabla \nabla T|^2. \quad (6.44)$$

The other terms may be estimated using Young's inequality, and we can derive

$$-2\text{Re}\{F^{i\bar{j}} \nabla_i |T|^2 \nabla_{\bar{j}} |\nabla T|^2\} \leq \frac{1}{2^3} |\nabla T|^4 + 2^6 \delta_2 |\nabla \nabla T|^2 + C |\nabla T|^3 + C |\nabla \text{Ric}_\omega|^3 + C.$$

Hence

$$\begin{aligned} \partial_t \{(|T|^2 + \tau_2) |\nabla T|^2\} & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F \{(|T|^2 + \tau_2) |\nabla T|^2\} - \frac{1}{8} |\nabla T|^4 \right. \\ & \quad \left. - \left(\frac{\tau_2}{4} - 2^6 \delta_2\right) |\nabla \nabla T|^2 + C |\nabla \text{Ric}_\omega|^3 + C |\nabla T|^3 + C \right\}. \end{aligned} \quad (6.45)$$

Combining (6.38) and (6.45) gives

$$\begin{aligned} \partial_t G & \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \left(\frac{1}{4} - 9\delta_1^2 - 9\tau_1\right) \alpha'^2 |\nabla \text{Ric}_\omega|^4 \right. \\ & \quad - \left(\frac{\tau_1}{4} - 2^8 \delta_1^2\right) (|\nabla \nabla \text{Ric}_\omega|^2 + |\nabla \bar{\nabla} \text{Ric}_\omega|^2) - \left(\frac{\tau_2}{4} - 2^6 \delta_2 - \delta_1^2 - \tau_1\right) |\nabla \nabla T|^2 \\ & \quad - \left(\frac{1}{8} - 2^4 3^4 \alpha'^{-2} (\delta_1^2 + \tau_1)^2 - 2^2 5^2 \delta_1^2\right) |\nabla T|^4 \\ & \quad \left. + C_{\alpha', \tau, \delta} |\nabla \text{Ric}_\omega|^3 + C_{\alpha', \tau, \delta} |\nabla T|^3 + C_{\alpha', \tau, \delta} \right\}. \end{aligned} \quad (6.46)$$

We may choose $\tau_1 = \min\{2^{-7}, 2^{-5} 3^{-2} |\alpha'|\}$ and $\tau_2 = 1$. Then for any $\delta_1, \delta_2 > 0$ such that

$$\delta_1, \delta_2 \leq 2^{-6} \tau_1 \ll \tau_2 = 1, \quad (6.47)$$

we have the estimate

$$\partial_t G \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \frac{1}{8} \alpha'^2 |\nabla \text{Ric}_\omega|^4 - \frac{1}{16} |\nabla T|^4 + C_{\alpha', \tau, \delta} \right\}. \quad (6.48)$$

Now, suppose G attains its maximum at a point (z, t) where $t > 0$. From the above estimate, at this point we have

$$\frac{1}{8} \alpha'^2 |\nabla \text{Ric}_\omega|^4 + \frac{1}{16} |\nabla T|^4 \leq C_{\alpha', \tau, \delta}. \quad (6.49)$$

It follows that G is uniformly bounded along the flow, and hence

$$|\nabla \text{Ric}_\omega| \leq C, \quad |\nabla T| \leq C, \quad (6.50)$$

along the flow.

Corollary 1 *There exists $0 < \delta_1, \delta_2$ with the following property. Suppose*

$$-\frac{1}{8} g^{p\bar{q}} < \alpha' \|\Omega\|^3 \bar{\rho}^{p\bar{q}} < \frac{1}{8} g^{p\bar{q}}, \quad (6.51)$$

$$|\alpha' \text{Ric}_\omega| \leq \delta_1, \quad (6.52)$$

and

$$|T|^2 \leq \delta_2, \quad (6.53)$$

along the flow. If there exists $\delta_0 > 0$ such that $0 < \delta_0 \leq \|\Omega\| \leq 1$ along the flow, then

$$|D^k \text{Ric}_\omega| \leq C, \quad |D^k T| \leq C, \quad (6.54)$$

where C depends only on $\delta_0, \delta_1, \delta_2, \alpha', \rho, \mu$, and $(X, \hat{\omega})$.

Proof: Since $\|\Omega\| = e^{-u}$, we are assuming that $|u|$ stays bounded, and that the metrics \hat{g} and $g = e^u \hat{g}$ are equivalent. We are also assuming that $e^{-u} |Du|_{\hat{g}}^2 \ll 1$ and $e^{-u} |\alpha' u_{\bar{k}j}|_{\hat{g}} \ll 1$. By Theorem 6, there exists δ_1 and δ_2 such that $|\nabla \nabla u|$ and $|\nabla \bar{\nabla} \nabla u|$ stay bounded along the flow. We will estimate partial derivatives in coordinate charts. Since

$$\partial_i \bar{\partial}_j \partial_k u = \nabla_i \bar{\nabla}_j \nabla_k u + \Gamma_{ik}^\lambda u_{\bar{j}\lambda}, \quad \partial_i \partial_j u = \nabla_i \nabla_j u + \Gamma_{ij}^\lambda u_\lambda, \quad (6.55)$$

and the Christoffel symbol

$$\Gamma_{ik}^\lambda = e^{-u} \hat{g}^{\lambda\bar{\gamma}} \partial_i (e^u \hat{g}_{\bar{\gamma}k}) = u_i \delta_k^\lambda + \hat{\Gamma}_{ik}^\lambda \quad (6.56)$$

stays bounded, we have that

$$|u|, |\partial u|, |\partial \partial u|, |\partial \bar{\partial} u|, |\partial \bar{\partial} \partial u| \leq C. \quad (6.57)$$

The scalar equation is

$$\partial_t u = \Delta_{\hat{\omega}} u + \alpha' e^{-2u} \tilde{\rho}^{p\bar{q}} u_{\bar{q}p} + \alpha' e^{-u} \hat{\sigma}_2(i\partial\bar{\partial}u) + |Du|_{\hat{\omega}}^2 + e^{-u}\nu. \quad (6.58)$$

where $\nu(x, u, Du)$. Differentiating once gives

$$\partial_t Du = \hat{F}^{p\bar{q}} Du_{\bar{q}p} + \alpha' D(e^{-2u} \tilde{\rho}^{p\bar{q}}) u_{\bar{q}p} + D|Du|_{\hat{g}}^2 - \alpha' e^{-u} \hat{\sigma}_2(i\partial\bar{\partial}u) Du + D(e^{-u}\nu), \quad (6.59)$$

where

$$\hat{F}^{p\bar{q}} = \hat{g}^{p\bar{q}} + \alpha' e^{-2u} \tilde{\rho}^{p\bar{q}} + \alpha' e^{-u} \hat{\sigma}_2^{p\bar{q}}. \quad (6.60)$$

We note that $\hat{F}^{j\bar{k}}$ only differs from $F^{j\bar{k}}$ (4.19) by a factor of e^u . From our assumptions on $|\alpha' \text{Ric}_{\omega}| = e^{-u} |\alpha' \partial\bar{\partial}u|_{\hat{g}}$ and $\|\Omega\| = e^{-u}$, we have uniform ellipticity of $\hat{F}^{j\bar{k}}$. Differentiating twice yields

$$\partial_t u_{\bar{k}j} = \hat{F}^{p\bar{q}} \partial_p \partial_{\bar{q}} u_{\bar{k}j} + \Psi(x, u, \partial u, \partial\bar{\partial}u, \partial\bar{\partial}\partial u), \quad (6.61)$$

where Ψ is uniformly bounded along the flow. By the Krylov-Safonov theorem [18], we have that $u_{\bar{k}j}$ is bounded in the $C^{\alpha/2, \alpha}$ norm. The function u and the spacial gradient Du are also bounded in the $C^{\alpha/2, \alpha}$ norm since the right-hand sides of (6.58) and (6.59) are bounded. We may now apply parabolic Schauder theory (for example, in [17]) to the linearized equation (6.59). Standard theory and a bootstrap argument give higher order estimates of u , and hence we obtain estimates on derivatives of the curvature and torsion of $g = e^u \hat{g}$.

7 Long time existence

Proposition 4 *There exists $M_0 \gg 1$ such that for all $M \geq M_0$, the following statement holds. If the flow exists on $[0, t_0)$, and initially starts with $u_0 = \log M$, then along the flow*

$$\frac{1}{C_1 M} \leq e^{-u} \leq \frac{C_2}{M}, \quad |T|^2 \leq \frac{C_3}{M^{4/3}}, \quad |\alpha' \text{Ric}_{\omega}| \leq \frac{1}{M^{1/2}}, \quad (7.1)$$

and

$$|D^k u|_{\hat{g}}^2 \leq \tilde{C}_k, \quad \frac{1}{2} \hat{g}^{j\bar{k}} \leq \hat{F}^{j\bar{k}} \leq 2 \hat{g}^{j\bar{k}}, \quad (7.2)$$

where \tilde{C}_k only depends on (X, \hat{g}) , μ , ρ , α' , M .

Proof: Let δ_1 and δ_2 be the constants from Corollary 1, and choose a smaller δ_1 if necessary to ensure $\delta_1 < 10^{-6}$. Recall that from Theorem 3,

$$\frac{1}{C_1 M} \leq \|\Omega\| = e^{-u} \leq \frac{C_2}{M} \quad (7.3)$$

along the flow for M large enough. Consider the set

$$I = \{t \in [0, t_0) \text{ such that } |\alpha' \text{Ric}_{\omega}| \leq \delta_1, \quad |T|^2 \leq \delta_2 \text{ holds on } [0, t]\}. \quad (7.4)$$

Since at $t = 0$ we have $|\alpha' \text{Ric}_\omega| = |T|^2 = 0$, we know that I is non-empty. By definition, I is relatively closed. We now show that I is open. Suppose $\hat{t} \in I$. By definition of I , the hypothesis of Theorem 4 is satisfied, hence $|T|^2 \leq \frac{C_3}{M^{\frac{4}{3}}} < \delta_2$ at \hat{t} as long as M is large enough. It follows that the hypothesis of Theorem 5 is satisfied as long as M is large enough, hence $|\alpha' \text{Ric}_\omega| \leq \frac{1}{M^{1/2}} < \delta_1$ at \hat{t} . We can conclude the existence of $\varepsilon > 0$ such that $[\hat{t} + \varepsilon) \subset I$, and hence I is open.

It follows that $I = [0, t_0)$. We know that $-C\hat{g}^{p\bar{q}} \leq \tilde{\rho}^{p\bar{q}} \leq C\hat{g}^{p\bar{q}}$ since $\tilde{\rho}$ can be bounded using the background metric. For M large enough, we can conclude

$$-\frac{1}{8}e^{-u}\hat{g}^{p\bar{q}} < \alpha'e^{-3u}\tilde{\rho}^{p\bar{q}} < \frac{1}{8}e^{-u}\hat{g}^{p\bar{q}}, \quad (7.5)$$

and we can apply Corollary 1 to obtain higher order estimates of u . Uniform ellipticity follows from the definition of $\hat{F}^{j\bar{k}}$ (6.60) and the estimates on $|\alpha' \text{Ric}_\omega| = e^{-u}|\alpha' \partial \bar{\partial} u|_{\hat{g}}$ and $\|\Omega\|$. Q.E.D.

Theorem 7 *There exists $M_0 \gg 1$ such that for all $M \geq M_0$, if the flow initially starts with $u_0 = \log M$, then the flow exists on $[0, \infty)$.*

Proof: By short-time existence [22], we know the flow exists for some maximal time interval $[0, T)$. If $T < \infty$, we may apply the previous proposition to extend the flow to $[0, T + \epsilon)$, which is a contradiction. Q.E.D.

8 Convergence of the flow

We may apply Theorem 7 to construct solutions to the Fu-Yau equation.

Theorem 8 *There exists $M_0 \gg 1$ such that for all $M \geq M_0$, if the flow initially starts with $u_0 = \log M$, then the flow exists on $[0, \infty)$ and converges smoothly to a function u_∞ , where u_∞ solves*

$$0 = i\partial\bar{\partial}(e^{u_\infty}\hat{\omega} - \alpha'e^{-u_\infty}\rho) + \frac{\alpha'}{2}i\partial\bar{\partial}u_\infty \wedge i\partial\bar{\partial}u_\infty + \mu, \quad \int_X e^{u_\infty} = M. \quad (8.1)$$

Proof: Since we will work with the scalar equation, all norms in this section will be with respect to the background metric $\hat{\omega}$. Let $v = \partial_t e^u$. Recall that

$$\int_X v = 0, \quad (8.2)$$

along the flow. Differentiating equation (2.28) with respect to time gives

$$2\partial_t v \frac{\hat{\omega}^2}{2!} = i\partial\bar{\partial}(v\hat{\omega} + \alpha'e^{-2u}v\rho) + \alpha'i\partial\bar{\partial}u \wedge i\partial\bar{\partial}(e^{-u}v). \quad (8.3)$$

Consider the functional

$$J(t) = \int_X v^2 \frac{\hat{\omega}^2}{2!}. \quad (8.4)$$

Compute

$$\begin{aligned} \frac{dJ}{dt} &= \int_X v i\partial\bar{\partial}(v\hat{\omega} + \alpha' e^{-2u} v\rho) + \alpha' \int_X v i\partial\bar{\partial}u \wedge i\partial\bar{\partial}(e^{-u}v) \\ &= - \int_X i\partial v \wedge \bar{\partial}v \wedge \hat{\omega} - \alpha' \int_X i\partial v \wedge \bar{\partial}(e^{-2u}v\rho) - \alpha' \int_X i\partial\bar{\partial}u \wedge i\partial v \wedge i\bar{\partial}(e^{-u}v) \\ &= - \int_X |\nabla v|^2 - \alpha' \int_X e^{-2u} i\partial v \wedge \bar{\partial}v \wedge \rho + 2\alpha' \int_X e^{-2u} v i\partial v \wedge \bar{\partial}u \wedge \rho \\ &\quad - \alpha' \int_X e^{-2u} v i\partial v \wedge \bar{\partial}\rho - \alpha' \int_X e^{-u} i\partial\bar{\partial}u \wedge i\partial v \wedge i\bar{\partial}v + \alpha' \int_X e^{-u} v i\partial\bar{\partial}u \wedge i\partial v \wedge i\bar{\partial}u. \end{aligned} \quad (8.5)$$

We may estimate

$$\begin{aligned} \frac{dJ}{dt} &\leq - \int_X |\nabla v|^2 + \alpha' \|\rho\| \int_X e^{-2u} |\nabla v|^2 + 2\alpha' \|\rho\| \|\nabla u\| \int_X e^{-2u} |v| |\nabla v| \\ &\quad + \alpha' \|\partial\rho\| \int_X e^{-2u} |v| |\nabla v| + \|\alpha' e^{-u} i\partial\bar{\partial}u\| \int_X |\nabla v|^2 \\ &\quad + \|\nabla u\| \|\alpha' e^{-u} i\partial\bar{\partial}u\| \int_X |v| |\nabla v|. \end{aligned} \quad (8.6)$$

By Proposition 4, we know that on $[0, \infty)$ we have the estimates

$$e^{-u} \leq \frac{C_2}{M} \ll 1, \quad |\nabla u|^2 \leq \frac{C_3 C_1}{M^{1/3}} \ll 1, \quad |\alpha' e^{-u} u_{\bar{k}j}| \leq \frac{1}{M^{1/2}}. \quad (8.7)$$

Hence for any $\varepsilon > 0$, we can choose M large enough such that

$$\frac{dJ}{dt} \leq -\frac{1}{2} \int_X |\nabla v|^2 + \varepsilon \int_X |v| |\nabla v| \leq -\left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \int_X |\nabla v|^2 + \frac{\varepsilon}{2} \int_X |v|^2. \quad (8.8)$$

Since $\int_X v = 0$, we may use the Poincaré inequality to obtain, for $\varepsilon > 0$ small enough,

$$\frac{dJ}{dt} \leq -\eta \int_X v^2 = -\eta J, \quad (8.9)$$

with $\eta > 0$. This implies that

$$J(t) \leq C e^{-\eta t}, \quad (8.10)$$

that is,

$$\int_X v^2 \leq C e^{-\eta t}. \quad (8.11)$$

From this estimate, we see that for any sequence $v(t_j)$ converging to v_∞ , we have $v_\infty = 0$.

We can now show convergence of the flow. Following the argument given in Proposition 2.2 in [2], we have

$$\begin{aligned}
\int_X |e^u(x, s') - e^u(x, s)| &\leq \int_X \int_s^{s'} |\partial_t e^u(x, t)| = \int_s^{s'} \int_X |v(x, t)| \\
&\leq \int_s^{s'} \left(\int_X v^2 \right)^{\frac{1}{2}} dt \leq \int_s^{+\infty} \left(\int_X v^2 \right)^{\frac{1}{2}} dt \\
&\leq C \int_s^{+\infty} e^{-\frac{\eta}{2}t} dt
\end{aligned} \tag{8.12}$$

Recall that we normalized the background metric such that $\int_X \frac{\hat{\omega}^2}{2} = 1$. This estimate shows that, as $t \rightarrow +\infty$, $e^u(x, t)$ are Cauchy in L^1 norm. Thus $e^u(x, t)$ converges in the L^1 norm to some function $e^{u_\infty}(x)$ as $t \rightarrow \infty$.

By our uniform estimates, e^{u_∞} is bounded in C^∞ , and a standard argument shows that e^u converges in C^∞ . Indeed, if there exist a sequence of times such that $\|e^{-u(x, t_j)} - e^{-u_\infty(x)}\|_{C^k} \geq \epsilon$, then by our estimates a subsequence converges in C^k to $e^{-u'_\infty}$. Then $\|e^{-u'_\infty(x)} - e^{-u_\infty(x)}\|_{L^1} = 0$ but $\|e^{-u'_\infty(x)} - e^{-u_\infty(x)}\|_{C^k} \geq \epsilon$, a contradiction.

It follows from (8.11) that e^{u_∞} satisfies the Fu-Yau equation (8.1).

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